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# Meditations on Ceva's Theorem

**Jürgen Richter-Gebert**
Technical University Munich
Zentrum Mathematik
Boltzmannstr. 3
D-85748 Garching
Germany

*This paper is dedicated to the unforgettable H. S. M. Coxeter, who had a striking ability to relate visual thinking to formal notions*

**Abstract.** This paper deals with the structure of incidence theorems in projective geometry. We will show that many of these incidence theorems can be interpreted as cyclic structures on a suitably chosen orientable manifold. Here the theorems of Ceva and Menelaus play the roles of basic building blocks for building larger theorems with greater complexity. In particular, we show how some other proofs for incidence theorems can be systematically translated into such "Ceva/Menelaus-proofs".

## 0 Introduction

> *Classical projective geometry was a beautiful field in mathematics. It died, in our opinion, not because it ran out of theorems to prove, but because it lacked organizing principles by which to select theorems that were important.*
>
> *R. MacPherson, M. McConnell, 1988* [17]

*A "proof" is something where many things come together and in the end everything closes up nicely to form a conclusion.* In this paper we will see that this very naïve view of mathematical proofs is almost literally correct for certain classes of geometric incidence theorems. We will show that by a certain pasting process many non-trivial incidence theorems can be generated by gluing many copies of Ceva and

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Menelaus configurations. These small building blocks are arranged at the faces of a manifold and the final conclusion of the theorem corresponds to the fact that the manifold is topologically closed.
The article deals with three different aspects of this construction principle.

- It will be shown how one can systematically generate incidence theorems by pasting together basic building blocks.
- We will see how a given incidence theorem can be analyzed and an underlying manifold structure can be revealed.
- It will be shown how to translate other proving techniques (in particular bi-quadratic final polynomials, as introduced in [5, 8, 18]) into Ceva/Menelaus proofs. For this translation process the theory of Tutte Groups, as introduced by Dress and Wenzel in a series of papers [9, 10, 11, 22] plays a decisive role.
- Finally, we will see how surgery on the manifolds, which underly the proofs, can be used, to generate even *spaces of theorems*.

This article is meant as an introduction to these fascinating interrelations between geometry, topology, combinatorics and algebra. We will, whenever possible, present pictures and diagrams to explain the basic concepts, rather than presenting purely formal notions.
To get a feeling of what this article is about, we start by reproducing a whole page of the famous book "Geometry Revisited" by Coxeter and Greitzer which appeared in 1967 [6]. The page contains a proof of Pappos's Theorem by a nice symmetric way of combining six Menelaus configurations that are substructures in Pappos's configuration. The pattern of the proof is (as we will see) as simple, as it is powerful: Try to locate sub-configurations corresponding to hypotheses such that each of them implies a relation of the form: $\frac{[...][...][...]}{[...][...][...]} = 1$. Multiply all the expressions. Perform an "orgy of cancellation" (Coxeter's words) and interpret what is left after the cancellation as the conclusion.
Proofs of this kind may be found all over the literature of projective incidence theory (already dating back to very old papers, such as those of Poncelet and Chasles at the very early days of projective geometry). This is not very surprising, since these kind of proofs are often the only ones that make an elegant use of the underlying algebraic structures. The aim of this paper is to give a systematic approach to this class of proofs.

## 1 Binomial Proofs

My personal starting point for the investigations in this article was a single nice proof for Pappos's Theorem and the desire to generalize the pattern of the proof to more general contexts. The original investigations were related to a systematic study of non-realizability proofs for arrangements of pseudolines (resp. oriented matroids) [5]. In the very last section of this article we will see an application of the more general theory in this context.

**1.1 Pappos's Theorem.** Let us start with Pappos's Theorem and the above mentioned proof of it. Pappos's Theorem is in a certain sense the smallest purely projective incidence theorem about points and lines. It states that, if one starts with two lines in the real projective plane and with three distinct points $1, 2, 3$ and $4, 5, 6$ on each line, then the three points $(1 \vee 5) \wedge (2 \vee 4)$, $(1 \vee 6) \wedge (3 \vee 4)$, $(2 \vee 6) \wedge (3 \vee 5)$ are

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*another, and if the three lines $AB$, $CD$, $EF$ meet $DE$, $FA$, $BC$, respectively, then the three points of intersection $L$, $M$, $N$ are collinear.*

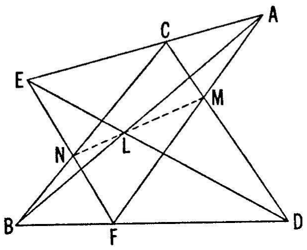



Figure 3.5A

The "projective" nature of this theorem is seen in the fact that it is a theorem of pure incidence, with no measurement of lengths or angles, and not even any reference to *order*: in each set of three collinear points it is immaterial which one lies between the other two. Figure 3.5A is one way of drawing the diagram, but Figure 3.5B is another, just as relevant. We can cyclically permute the letters $A$, $B$, $C$, $D$, $E$, $F$, provided we suitably re-name $L$, $M$, $N$. To avoid considering points at infinity, which would take us too far in the direction of projective geometry, let us assume that the three lines $AB$, $CD$, $EF$ form a triangle $UVW$, as in Figure 3.5C. Applying Menelaus's theorem to the five triads of points

$$LDE, \quad AMF, \quad BCN, \quad ACE, \quad BDF$$

on the sides of this triangle $UVW$, we obtain

$$\frac{VL}{LW}\frac{WD}{DU}\frac{UE}{EV} = -1, \quad \frac{VA}{AW}\frac{WM}{MU}\frac{UF}{FV} = -1, \quad \frac{VB}{BW}\frac{WC}{CU}\frac{UN}{NV} = -1,$$

$$\frac{VA}{AW}\frac{WC}{CU}\frac{UE}{EV} = -1, \quad \frac{VB}{BW}\frac{WD}{DU}\frac{UF}{FV} = -1.$$

Dividing the product of the first three expressions by the product of the last two, and indulging in a veritable orgy of cancellation, we obtain

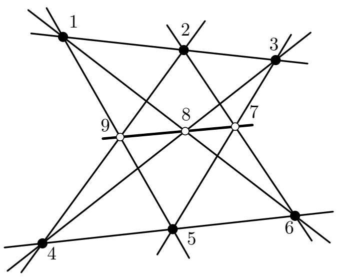
$$\frac{VL}{LW}\frac{WM}{MU}\frac{UN}{NV} = -1,$$

whence $L$, $M$, $N$ are collinear, as desired. [17, p. 237.]

Coxeter/Greitzer's proof of Pappos's Theorem.

automatically collinear as well. There are many different proofs for this fundamental theorem. We will consider a particularly well structured one. The theorem could be restated in the following way: *If in* $\mathbb{RP}^2$ *the triples of points* $(1,2,3)$, $(1,5,9)$, $(1,6,8)$, $(2,4,9)$, $(2,6,7)$, $(3,4,8)$, $(3,5,7)$, $(4,5,6)$ *are collinear, then* $(7,8,9)$ *is collinear as well.*

We give a proof of the theorem under the additional non-degeneracy assumptions that no two lines of the picture coincide and that furthermore $(1,4,7)$ is not collinear.




| | | | |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | $a$ | $b$ | $c$ |
| 3 | $d$ | $e$ | $f$ |
| 4 | 0 | 1 | 0 |
| 5 | $g$ | $h$ | $i$ |
| 6 | $j$ | $k$ | $l$ |
| 7 | 0 | 0 | 1 |
| 8 | $m$ | $n$ | $o$ |
| 9 | $p$ | $q$ | $r$ |

| | | |
|---|---|---|
| $(1,2,3)$ | $\Longrightarrow$ | $ce{=}bf$ |
| $(1,5,9)$ | $\Longrightarrow$ | $iq{=}hr$ |
| $(1,6,8)$ | $\Longrightarrow$ | $ko{=}ln$ |
| $(2,4,9)$ | $\Longrightarrow$ | $ar{=}cp$ |
| $(2,6,7)$ | $\Longrightarrow$ | $bj{=}ak$ |
| $(3,4,8)$ | $\Longrightarrow$ | $fm{=}do$ |
| $(3,5,7)$ | $\Longrightarrow$ | $dh{=}eg$ |
| $(4,5,6)$ | $\Longrightarrow$ | $gl{=}ij$ |
| $(7,8,9)$ | $\Longleftarrow$ | $mq{=}np$ |

Pappos's Theorem, the smallest projective theorem.

**Proof** Assume that the points are given by homogeneous coordinates as indicated by the matrix above. By the assumption that $(1,4,7)$ is not collinear we can in particular assume that (if necessary, after a suitable projective transformation) the points 1, 4 and 7 are represented by the three unit vectors. With this special choice of the coordinates, each of the eight collinearities of the hypotheses can be expressed as the vanishing of a certain $2 \times 2$ sub-determinant of the coordinate matrix. If we write down all these equations (compare picture above), multiply all left sides and multiply all right sides we are left with another equation $mq = np$ which translates back to the collinearity of $(7,8,9)$. By our non-degeneracy assumptions all variables involved in the proof will be non-zero, therefore the cancellation process is feasible. □

The non-degeneracy assumptions made for the proof may at first sight seem a bit unnatural. In fact, Pappos's Theorem stays valid also in the absence of these assumptions. However, for most more complicated theorems additional non-degeneracy assumptions are unavoidable (compare [18]). If not otherwise stated, for the rest of the paper, we will always assume that different lines of a configuration that share a common configuration point will not coincide. This ensures in general that during the cancellation process in our proofs we will never divide by "0".

**1.2 The general pattern.** If one tries to generalize the proof structure of the previous section, one encounters unexpected difficulties. The proof heavily relies on the very special choice of the basis. This choice made it possible to express every collinearity as a $2 \times 2$ determinant, since every collinear triple contained at least one basis point. Already for simple theorems like Desargues's Theorem (which consists of 10 points and 10 lines) such a special choice is no longer possible. No matter how we choose a basis, there will always be one collinear triple that does not contain a basis point and therefore must be expressed by the vanishing of a $3 \times 3$ determinant.

The key observation that helps to bypass this dilemma is the fact that the equations of the form, $XY = WZ$ can be interpreted as a certain fragment of a suitably chosen Grassmann Plücker relation. Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^3$ be the homogeneous coordinates of points in $\mathbb{RP}^2$. We abbreviate $[a,b,c] := \det(x_a, x_b, x_c)$. For sake of simplicity we will identify a point with its index.

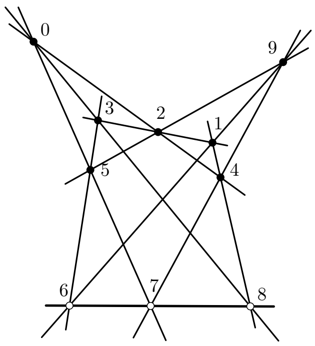
It is a well-known fact that for arbitrary points $a,b,c,d,e \in \mathbb{RP}^2$ we have

$$[a,b,c][a,d,e] \;-\; [a,b,d][a,c,e] \;+\; [a,b,e][a,c,d] \;=\; 0.$$

If in addition $(a, b, c)$ is collinear, then we have automatically $[a, b, c] = 0$ and this, together with the Grassmann Plücker relation, implies the binomial equation

$$[a, b, d][a, c, e] \;=\; [a, b, e][a, c, d].$$

Conversely, if we know that this equation holds, we can conclude that either $(a, b, c)$ or $(a, d, e)$ is collinear since the first term of the Grassmann Plücker relation then has to vanish.
In principle, the equation $[a, b, d][a, c, e] = [a, b, e][a, c, d]$ may be considered as a kind of coordinate free version of a $2 \times 2$ determinant. We can use this interpretation directly to generate very well structured proofs for projective incidence theorems. We will demonstrate this technique (so called *binomial proofs*) by two examples.
First consider Desargues's Theorem (shown in the next picture). If (in $\mathbb{RP}^2$) all collinearities except $(7, 6, 8)$ are present as in the picture and no two of the lines coincide then $(7, 6, 8)$ is also automatically collinear.

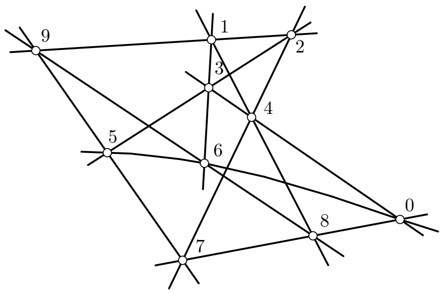



| | | |
|---|---|---|
| (479) | $\Longrightarrow$ | [471][496] = [476][491] |
| (916) | $\Longrightarrow$ | [914][962] = [912][964] |
| (259) | $\Longrightarrow$ | [256][291] = [251][296] |
| (240) | $\Longrightarrow$ | [248][203] = [243][208] |
| (083) | $\Longrightarrow$ | [082][035] = [085][032] |
| (570) | $\Longrightarrow$ | [573][508] = [578][503] |
| (213) | $\Longrightarrow$ | [215][234] = [214][235] |
| (418) | $\Longrightarrow$ | [412][487] = [417][482] |
| (536) | $\Longrightarrow$ | [532][567] = [537][562] |
| (768) or (745) | $\Longleftarrow$ | [764][785] = [765][784] |

Desargues's configuration . . . . . . and its proof

**Proof** For the proof consider the equations to the right of the picture. The first 9 equations are consequences of the 9 hypotheses of the theorem. If we multiply all left sides and all right sides and cancel terms that occur on both sides, we are left with the last equation (the cancellation process is possible since all determinants that occur in the equations are non-zero by our non-degeneracy assumptions). By the Grassmann Plücker argument the last equation implies that either $(7, 6, 8)$ or $(7, 4, 5)$ is collinear. Since the non-collinearity of $(7, 4, 5)$ is among our non-degeneracy assumptions, we can conclude that the triple $(6, 7, 8)$ has to be collinear. $\square$

As a final example of this kind let us consider the configuration shown in the next picture. It shows a certain $10_3$-configuration (10 points, 10 lines, and three points on each line) that has the property that it is geometrically not realizable without additional degeneracies (observe that the line $(5, 6, 0)$ is slightly bent). The equations on the right of the picture demonstrate that, if all 10 collinearities are satisfied as indicated in the picture, then (by the usual cancellation argument) we can conclude that also $[578][670] = [570][678]$ holds. This however implies that either $(5, 1, 2)$ or $(5, 7, 0)$ is collinear. Both cases force a massive degeneration of the configuration.




$$
\begin{array}{rcl}
(129) & \Longrightarrow & [128][179] = +[127][189] \\
(136) & \Longrightarrow & [146][130] = -[134][160] \\
(148) & \Longrightarrow & [124][168] = -[128][146] \\
(235) & \Longrightarrow & [234][250] = -[245][230] \\
(247) & \Longrightarrow & [127][245] = -[124][257] \\
(304) & \Longrightarrow & [134][230] = +[130][234] \\
(056) & \Longrightarrow & [160][570] = +[150][670] \\
(759) & \Longrightarrow & [157][789] = -[179][578] \\
(869) & \Longrightarrow & [189][678] = -[168][789] \\
(780) & \Longrightarrow & [578][670] = +[570][678] \\
\hline
 & & [157][250] = +[150][257]
\end{array}
$$

A non-realizable $10_3$ configuration.

**1.3 Automatic proving.** The above proving technique was successfully applied in [8, 18] to create algorithms that prove many theorems in projective geometry automatically. The algorithm has the particularly nice feature that, if it finds a proof, the proof admits a clear readable structure, such that its correctness can be checked by hand easily. This is an interesting contrast to automatic proving techniques that are based on methods of commutative algebra (like Gröbner Basis or Ritt's algebraic decomposition method) that produce proofs consisting of large polynomials of generally high degree.

The basic idea of such a binomial-based proving algorithm is simple. First one generates all binomial relations that are consequences of the hypotheses of the theorem. Then one creates binomial expressions that imply the conclusion and finally tests whether one of the conclusion binomials can be generated as a suitable combination of the hypotheses binomials. In fact, this last step can be carried out, in principle, by a linear equation solver, since one is interested in linear combinations of the exponent vectors. In this approach the determinants themselves are treated as formal symbols (variables) and one has never to go down to the concrete level of coordinates.

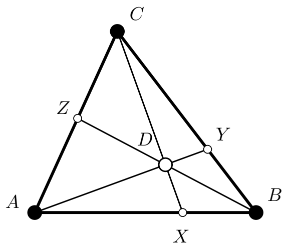
Although this method has the potential to find nice and well structured proofs of this kind, if they exist, it has a great disadvantage. When in the third step the linear equation solver searches for a suitable dependence it has "forgotten" all structural information about the theorem. In essence it searches "blindly" for a linear dependence in a space in which the variables correspond to the $\binom{n}{3}$ determinants, and where each collinearity corresponds to many binomial equations. Since the calculations all have to be carried out in exact arithmetic, one may easily run into space or time problems. It would be much more desirable to have some insight in the possible structures of such proofs to rule out many unreasonable cancellation patterns in advance. This is what the rest of this paper is about.

## 2 Theorems on manifolds

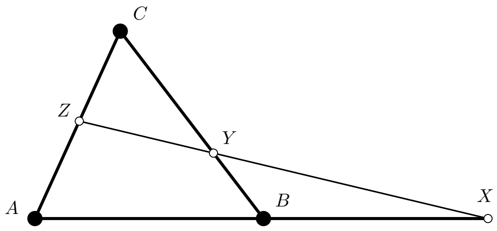
In this section we will give a different view on cancellation patterns that can be used to prove incidence theorems. The cancellation pattern used now will have the additional feature that it can be interpreted directly in a topological way.

**2.1 The theorems of Ceva and Menelaus.** We will sketch a remarkable relation between incidence theorems and cycles on manifolds. At first sight the presented approach to incidence theorems seems to be very special but indeed we will demonstrate in Section 6 that this approach is as expressive as the binomial proofs described in the previous section.

Our main protagonists are the theorems of Ceva and of Menelaus. Ceva's Theorem states that if in a triangle the sides are cut by three concurrent lines that pass through the corresponding opposite vertex, then the product of the three (oriented) length ratios along each side equals 1. Menelaus's Theorem states that this product is $-1$ if the cuts along the sides come from a single line.




Ceva's Thm: $\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1$




Menelaus's Thm: $\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = -1$

In fact, these theorems are almost trivial if one views the length ratios as ratios of certain triangle areas. For this observe that, if the line $(A, B)$ is cut by the line $(C, D)$ at a point $X$, then we have

$$\frac{|AX|}{|XB|} = -\frac{\Delta(C, D, A)}{\Delta(C, D, B)}, \qquad (*)$$

where $\Delta(A, B, C)$ denotes the oriented triangle area.

In order to prove Ceva's Theorem we consider the obvious identity:

$$\frac{\Delta(CDA)}{\Delta(CDB)} \cdot \frac{\Delta(ADB)}{\Delta(ADC)} \cdot \frac{\Delta(BDC)}{\Delta(BDA)} = -1,$$

(note that the triangle area $\Delta$ is an alternating function and that each triangle in the denominator occurs as well in the numerator). Applying the above identity $(*)$ we immediately get Ceva's Theorem. Similarly, a proof of Menelaus's Theorem is derived. For this consider the special line as being generated by two points $D$ and $E$. We have

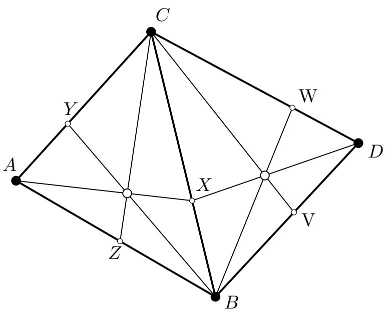
$$\frac{\Delta(DEA)}{\Delta(DEB)} \cdot \frac{\Delta(DEB)}{\Delta(DEC)} \cdot \frac{\Delta(DEC)}{\Delta(DEA)} = 1,$$

Applying the identity $(*)$ yields Menelaus's Theorem. Observe that the expressions of Ceva and Menelaus carry an orientation information. If in the future we talk about the "Ceva-expression" (or "Menelaus-expression") for the triangle $A, B, C$ the letters $A$, $B$, and $C$ are assumed to be ordered as in the expression above. $A, C, B$ would generate the reciprocal expression. Furthermore, we will call the points $X$, $Y$, and $Z$ in the above drawing the *edge points* of the configuration. The points $A$, $B$, and $C$ will be called the *vertices* of the configuration. Point $D$ in Ceva's configuration will be called the *Ceva point* and the cutting line in Menelaus's configuration is the *Menelaus line*.

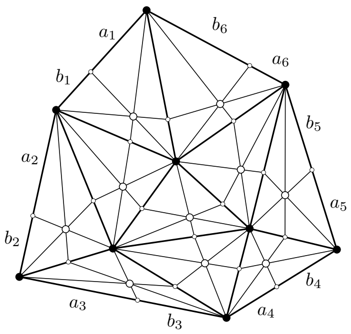
**2.2 A homotopy argument.** Now, consider the situation where two triangles that are equipped with a Ceva configuration share an edge and the corresponding edge point on this edge (see the picture below). The triangle $A, B, C$ yields a relation $\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1$ while the triangle $C, B, D$ yields $\frac{|CX|}{|XB|} \cdot \frac{|BV|}{|VD|} \cdot \frac{|DW|}{|YW|} = 1$. The quotient $\frac{|BX|}{|XC|}$ occurs in the first expression and its reciprocal occurs in the second expression. If we multiply both expressions, this quotient cancels and we are left only with terms that live on the boundary of the figure. We obtain

$$\frac{|AZ|}{|ZB|} \cdot \frac{|CY|}{|YA|} \cdot \frac{|BV|}{|VD|} \cdot \frac{|DW|}{|YW|} = 1.$$

We now consider a triangulated topological disc. All triangles of the triangulation should be equipped with Ceva configurations that have the additional property that points on interior edges are the shared edge points of the two adjacent triangles. We consider the product of all corresponding Ceva-expressions.




Gluing two Ceva Configutarions



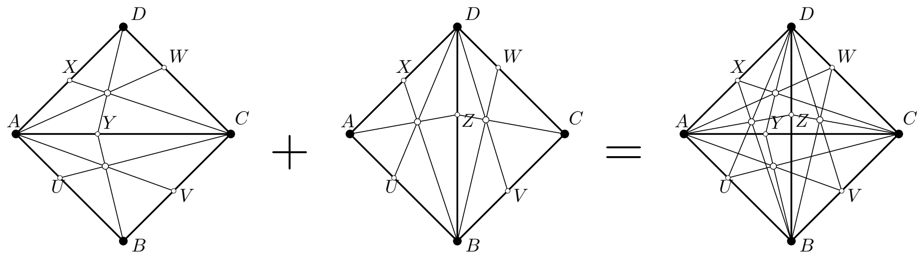

Gluing many Ceva Configutarions

If the triangles are oriented consistently (adjacent triangles use the common edge in opposite directions), all quotients related to inner edges will cancel. We are left with an expression that only depends on the position of the boundary points (including the edge points along the boundary edges). If in the last picture on the right the letters $a_1, b_1, , \ldots, a_6, b_6$ correspond to the oriented lengths around the boundary we can conclude immediately that we must have

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} \quad = \quad 1.$$

Now, consider any triangulated manifold that forms an oriented 2-cycle. This cycle serves as a kind of *frame* for the construction of an incidence theorem. It is important to mention in what category we understand the term "triangulated manifold". We consider compact, orientable 2 manifolds without boundary and subdivisions by CW-complexes whose faces are triangles. So in principle, already a subdivision of a 2-sphere by two topological triangles, which are identified along the edges, would be a feasible object for our considerations.
Consider such a cycle as being realized by flat triangles (it does not matter if these triangles intersect, coincide or are coplanar as long as they represent the combinatorial structure of the cycle). By the above argument the presence of Ceva configurations on all but one of the faces will imply automatically the existence of a Ceva configuration on the final face. Thus at the final face the three lines

connecting the edge points and the vertices will meet automatically, and we have an incidence theorem. In what follows we will study many concrete examples of this amazingly rich construction technique.
As a first example take the projection of a tetrahedron $(ABCD)$ to $\mathbb{R}^2$. Now, choose Points $U, V, W, X, Y, Z$ one on each of the edges of the tetrahedron. Assume that for three of the faces these points form a Ceva configuration. Then they automatically form a Ceva configuration on the last face — an incidence theorem.




Although the proof of this incidence theorem is already evident by our above homotopy argument, we still want to present the algebraic cancellation pattern in detail. Consider the following formula

$$\begin{aligned}\left(\frac{|AU|}{|UB|}\cdot\frac{|BV|}{|VC|}\cdot\frac{|CY|}{|YA|}\right)\cdot\left(\frac{|CW|}{|WD|}\cdot\frac{|DX|}{|XA|}\cdot\frac{|AY|}{|YC|}\right)\cdot&\\ \left(\frac{|AX|}{|XD|}\cdot\frac{|DZ|}{|ZB|}\cdot\frac{|BU|}{|UA|}\right)\cdot\left(\frac{|BZ|}{|ZD|}\cdot\frac{|DW|}{|WC|}\cdot\frac{|CV|}{|VB|}\right)&\quad=\quad 1.\end{aligned}$$

This formula is obviously true, since all lengths of the numerator occur in the denominator as well and vice versa (this property is inherited from the cyclic structure). On the other hand, each of the factors in brackets being 1 states the Ceva condition for one of the faces. Thus three of these conditions imply the last one. The essential fact that makes this proof work is that whenever two faces meet in an edge the two corresponding ratios cancel. In general we obtain:

> *For any triangulated oriented 2-CW-cycle choose a point on each edge such that for every face either a Ceva or a Menelaus condition is generated. If altogether an even number of Menelaus configurations is involved, then the conditions on all but one of the triangles automatically imply the condition on the last triangle.*

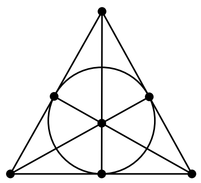
We need an even number of Menelaus configuration since each Menelaus configuration accounts for a factor of $-1$ in the product. We will call such a cycle equipped with Ceva/Menelaus configurations a Ceva/Menelaus-cycle. Instead of drawing the whole incidence structure we often simply draw a schematic diagram, in which we indicate the combinatorial structure of the cycle and attach a label $C$ or $M$ to each of the faces. We will usually draw these schemes as a planar net of triangles for which we specify which vertices and which edges have to be identified.

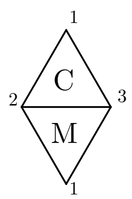
## 3 A census of incidence theorems

At first instance the method described in the last section is very useful for producing geometric incidence theorems by pasting together triangles that carry Ceva or Menelaus configurations. In this chapter we want to elaborate on this aspect.

We will at least for small numbers of triangles list several examples that can be produced by this philosophy. In any case we need a configuration of triangles that forms a closed orientable CW-cycle. It may happen that two triangles are identified along more than one edge. However, to avoid trivial cases (in which the two configurations of these triangles together with the edge points simply coincide), we have to assume that in such a case one triangle is equipped with a Ceva configuration and the other with a Menelaus configuration. Later on such a sub-configuration consisting of two triangles that coincide along two edges will be called a “pocket”. Let us now start with a census of small incidence theorems. Observe that the number of triangles involved in a Ceva/Menelaus-cycle must be even.

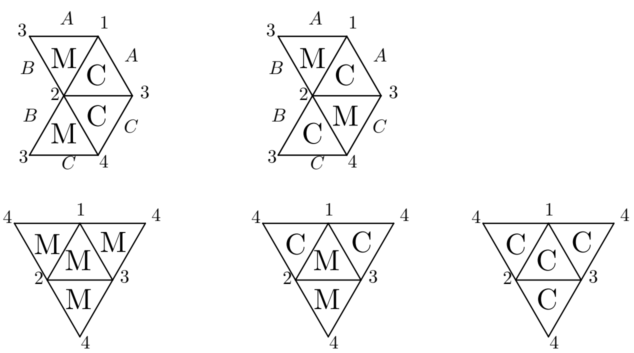
**3.1 Two triangles.** For two triangles there is only one possibility of forming a Ceva/Menalaus-cycle. We can only identify the two triangles along their edges. Then we have to assign to one of the triangles a Ceva configuration and to the other one a Menelaus configuration. In the real projective plane this configuration is not realizable at all, since one triangle forces the product of ratios to be $-1$ the other forces the product of ratios do be 1. However, if we consider a field of characteristic two, then this configuration is an incidence theorem. In fact, this configuration is nothing else but the well-known Fano plane. In the triangle scheme below the labels at the vertices of the triangles indicate which vertices have to be identified.






The Fano configuration . . . . . . and its triangle scheme

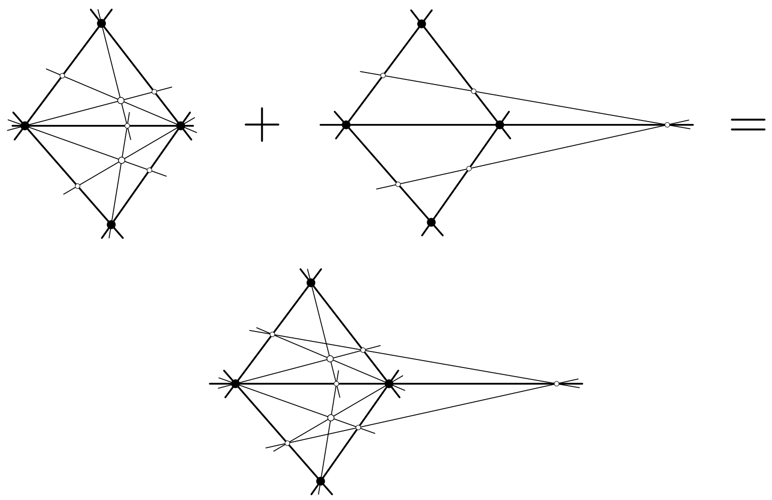
**3.2 Four triangles.** The structure becomes considerably richer if we consider four instead of only two triangles. First of all there are two combinatorially different ways of creating a manifold involving four triangles: either they could form a tetrahedron, or they could form a manifold, in which two *pockets* formed by two triangles are pasted. It is important to observe that for the last case one has not only to make clear which vertices have to be identified, but also which edges (and with them the edge points) have to be identified.




All five possibilities to form an incidence theorem on four triangles.

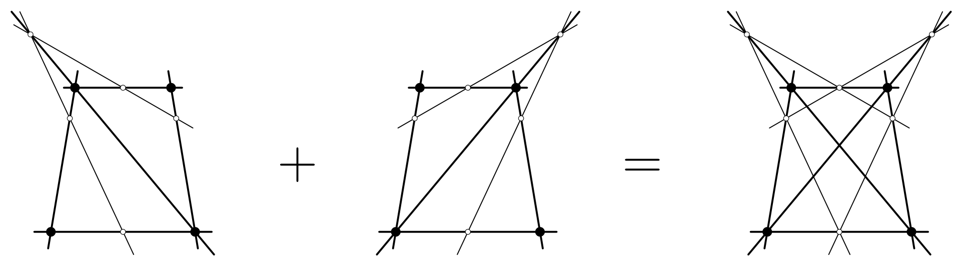
The first row shows the two possible cases for the "pocket" version. The second row shows the three possibilities of assigning Ceva and Menelaus configurations to a tetrahedron.

Let us now analyze the geometric interpretation of these cases as incidence theorems. The situation for the first example of the first row is shown in the next picture. In principle, two adjacent triangles that carry a Ceva configuration are overlaid with two adjacent triangles that carry Menelaus configurations. The final coincidence of the resulting configuration is satisfied automatically. The resulting image is the well-known configuration that shows that for three given points on a line one can construct a harmonic point by erecting a drawing of a tetrahedron over the line.



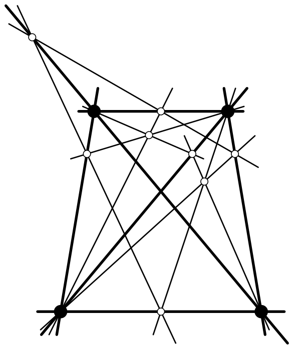
Although the second picture in the first row is combinatorially different from the first one, the resulting incidence configuration is exactly the same. The patient reader is invited to check this.

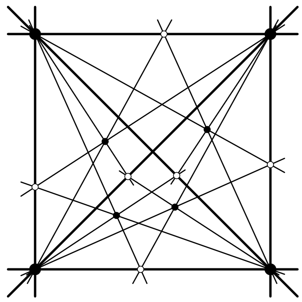
The first scheme in the second row is nothing else but a description of Desargues's Theorem, which we already encountered in Section 1. The following picture shows the decomposition of the tetrahedron in a front and a back part.



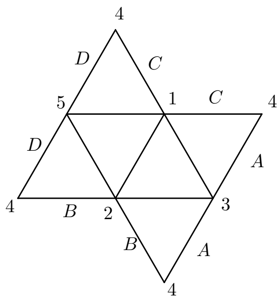
The last two theorems in the second row correspond to the incidence configurations shown in the next picture. In fact, the last example is the one we already have studied when we introduced the proving technique three pages earlier. This configuration has a few remarkable special cases in which one or more points do coincide.

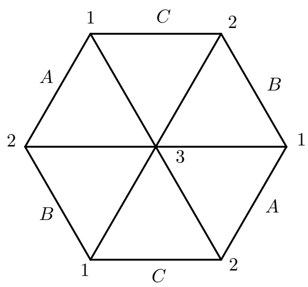
However, we will not analyze them here in detail.





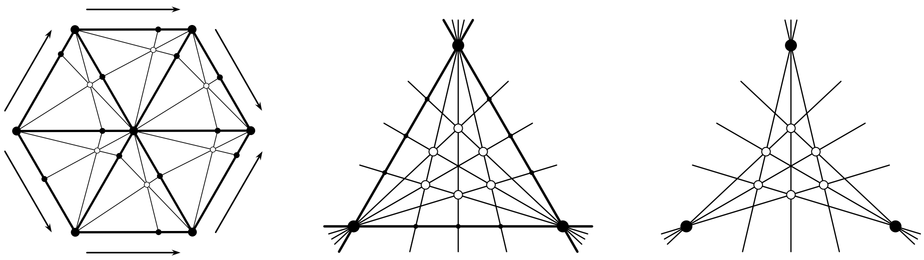
**3.3 Six triangles.** Already with six triangles the situation becomes quite elaborate. There are altogether six combinatorially different types of cycles, and 21 different types of underlying incidence configurations. We will not list all of them here. Some of the manifolds involve "pocket" sub-configurations and will be considered in a more general setting later on in Section 4. The only cycles that do not contain pockets are shown in the next picture. The cycle on the left is a double pyramid over a triangle. There are 8 non-isomorphic ways to equip it with Ceva and an even number of Menelaus configurations. We will not study these cases here. The second cycle is topologically more interesting. It is the smallest CW-decomposition of a torus into triangles. It has only three vertices and nine edges. There are exactly four ways of assigning a Ceva/Menelaus-cycle to it: all triangles Ceva, exactly two adjacent triangles Menelaus, exactly two adjacent triangles Ceva, all triangles Menelaus. The next subsection will be dedicated to a detailed discussion of the first and the last one of these situations.



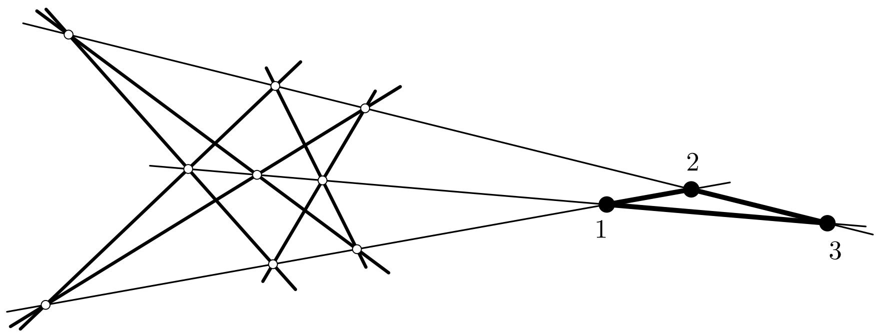

**3.4 Pappos revisited.** We now will deal exclusively with the situation of six triangles that form a topological torus. First observe, that if we realize this cycle by flat triangles, then the only way of doing this is by making all triangles coincident. Assume the triangles are realized in this way and furthermore assume that we assign a Ceva configuration to each of these triangles. The left drawing of the next picture shows the unfolded configuration, while the middle drawing shows the overlay. Observe that for each of the triangles we get one Ceva point. Lines of two adjacent triangles that share the corresponding edge point become identified. So, all together we have the three original points of the triangle, and six

Ceva points. These nine points together with the nine interior lines have exactly the combinatorics of Pappos's Theorem. Thus we have obtained a "Ceva-proof" for Pappos's Theorem. It is an amazing fact that by the pairwise identifications of the interior lines the edge points and also the original edges of the triangles do not play any role in the theorem we just proved. The edge points are just intersections of two lines. After deleting these points, the edges of the triangles are just joins of two points and can be also deleted. The picture on the right shows the situation after the deletion. It is just a nice and symmetric drawing of Pappos's Theorem.



A Ceva-proof for Pappos's Theorem

It is a really surprising fact that also in the case in which we assign to all six triangles a Menelaus configuration we get a drawing of Pappos's Theorem. The next picture shows a drawing of this configuration. The original triangle of the manifold is the dark triangle in the right of the drawing. The additional six lines in the drawing are the six Menelaus lines that cut all three edges of the triangles. The nine points on the left of the configuration correspond to the nine edges of the triangles in our torus. Similar to the fact that in the last proof the edges of the triangles could be neglected, this time, the three vertices are superfluous. This way of decomposing Pappos's Theorem into six Menelaus configurations was exactly the method we encountered in the original copy of Coxeter's page shown in Section 1.




A Menelaus-proof for Pappos's Theorem

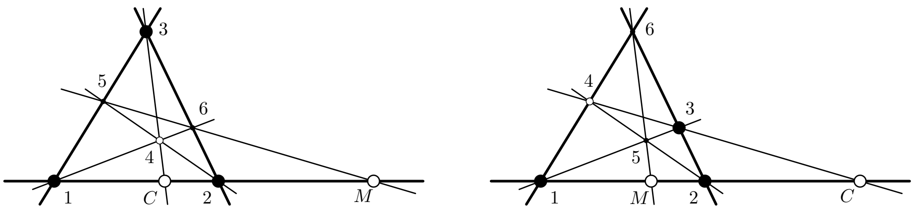
## 4 Basic building blocks

In this section we want to review a few basic substructures that are very often present in Ceva/Menelaus-proofs. In general, these basic substructures can be directly assigned to elementary principles of projective geometry – like cross ratios, harmonic points, or perspectives.

**4.1 Pockets, ….** In the last section we already encountered the situation in which two triangles were joined along *two* edges. In this case it was necessary to equip one of the triangles with a Ceva configuration and the other with a Menelaus configuration (otherwise the third edge-point would coincide as well). In Section 3.1 we already saw that we can identify the two triangles along the third line only in fields of characteristic 2. Here we study the case, in which we do not identify the edge points on the third line. In this case we will have exactly four points on this line: two original triangle edges, one Ceva-edge-point and one Menelaus-edge-point. The next picture (left) shows the corresponding incidence configuration. Topologically we can consider the pocket as a disk that is bounded by two edges $1 - C - 2$ and $2 - M - 1$. If we multiply the Ceva and the Menelaus condition we immediately get

$$\frac{|1C|}{|C2|} \cdot \frac{|2M|}{|M1|} = -1.$$

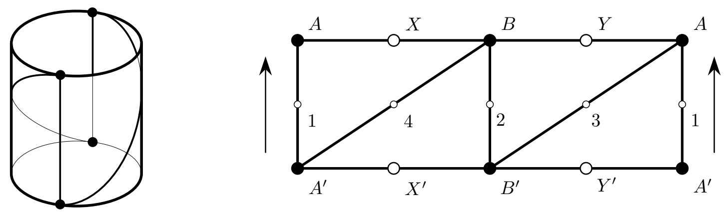
In other words $1, 2, C$ and $M$ are in harmonic position.




Accidentally, we can interpret exactly the same incidence configuration also the other way around. If we interchange the role of the points $3, \ldots, 6$ according to $3 \leftrightarrow 6$, $4 \leftrightarrow 5$, $5 \leftrightarrow 4$, $6 \leftrightarrow 3$, we see that for the (old) triangle $1, 2, 6$ we obtain an interchanged role of the Ceva and the Menelaus edge point. This shows that if we have a pocket in a Ceva/Menelaus-cycle, we can freely interchange the role of the "$C$" and the "$M$" on the pocket without changing the incidence configuration.

**4.2** … **tunnels, ….** A very common situation in projective incidence theorems is that "information" about a certain substructure is transfered from one part of the configuration to another by projection. For instance, if four points on a line have a certain cross ratio, any projection of these points onto another line will result in four points with the same cross ratio. We can perfectly model this situation by suitable substructures of Ceva/Menelaus-cycles. Assume that we have a CW-triangulation of a manifold that is homeomorphic to $S^1 \times [0, 1]$ (a sphere with two holes). The boundary of this manifold consists of two disjoint circles. If each triangle is equipped with a Ceva or Menelaus configuration (such that the number of Menelaus configuration is even), then the product of the length ratios along the first cycle must be identical to the product of the length ratios on the second (or its

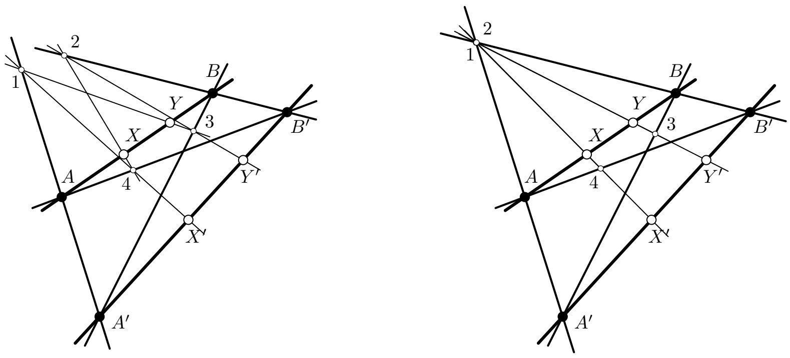
inverse, depending on the orientation). Thus in a sense the configuration transfers information from the first boundary cycle to the second one.
The smallest possible situation is shown in the next picture. It consists of 4 triangles that form a simple cycle. Combinatorially the situation is a tetrahedron, with two cuts along two opposite edges. The cycles consist just of two edges, each.




If the situation is as in the picture, then for the corresponding products of length ratios we get:

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YA|} \quad = \quad \frac{|A'X'|}{|X'B'|} \cdot \frac{|B'Y'|}{|Y'A'|}.$$

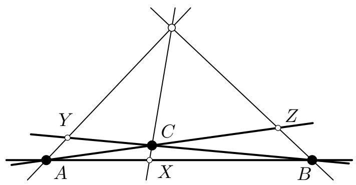
In other words, the configuration transfers the cross ration of $A, B, X, Y$ to the points $A', B', X', Y'$. If all triangles are equipped with Menelaus's configurations, the corresponding incidence configuration is shown in the following picture (left).

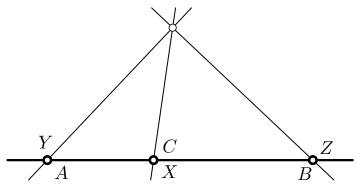



The right picture shows a special instance of this configuration, in which the points 1 and 2 (and several lines) coincide. In this picture the points $A, B, X, Y$ and the points $A', B', X', Y'$ are connected to each other by a perspectivity. So we obtained a Ceva/Menelaus-proof for the fact that the cross ratios are preserved by perspectivities.

**4.3** ... **and toothpicks.** In this subsection we will study the semantics of degenerate Ceva triangles and degenerate Menelaus triangles. Usually, a Ceva configuration produces a relation $\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1$. This relation remains also valid if the points $A, B, C$ are collinear. In this case, however, the points $X, Y, Z$ will coincide with the vertices. And we will get:

$$\frac{|AC|}{|CB|} \cdot \frac{|BA|}{|AC|} \cdot \frac{|CB|}{|BA|} = 1.$$




Similarly, applying Menelaus's Theorem to a degenerate triangle $A, B, C$ we see that the Menelaus line intersects the line $A, B, C$ at a unique point $X$, since all edge points will coincide. In this case we get the identity

$$\frac{|AX|}{|XB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CX|}{|XA|} = -1.$$

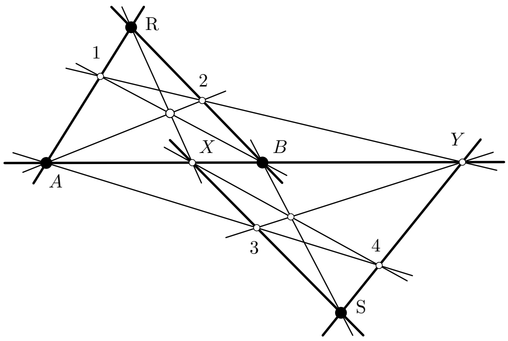
Here $X$ can be any point on the line through $A, B, C$.
One might wonder whether such degenerate configurations may ever occur in a real proof of an incidence theorem. From the point of view of Ceva/Menelaus proofs the length ratios $\frac{|AC|}{|CB|}$ must be considered as indecomposable units. The degenerate subconfiguration gives us the possibility to replace $\frac{|AC|}{|CB|}$ by the product $\frac{|CA|}{|AB|} \cdot \frac{|AB|}{|BC|}$. In particular, this gives us the possibility to exchange the role of edge points and vertices. In fact, allowing degenerate triangles considerably broadens the applicability of our proving methods. We want to discuss briefly how to derive a trivial (but useful) identity in a Ceva/Menelaus setup. Assume that $A, B, X, Y$ are four points on a line. Then we obviously have:

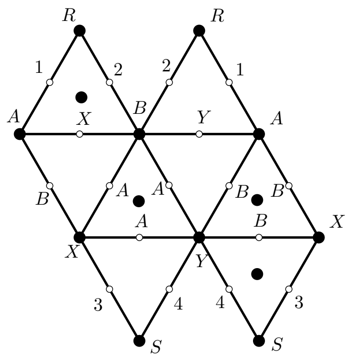
$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YA|} = \frac{|BY|}{|XB|} \cdot \frac{|AX|}{|YA|}.$$

However, if we consider the four length ratios as unbreakable symbols the identity is far from being trivial. We can establish this identity in our setup by pasting two degenerate Ceva configurations (triangles $A, B, X$ and $A, B, Y$) and two degenerate Menelaus configurations (triangles $X, Y, A$ and $X, Y, B$). The following calculation proofs the identity:

$$\begin{aligned} &\left(\underline{\tfrac{|AX|}{|XB|}} \cdot \tfrac{|BA|}{|AX|} \cdot \tfrac{|XB|}{|BA|}\right) \cdot \left(\underline{\tfrac{|BY|}{|YA|}} \cdot \tfrac{|AB|}{|BY|} \cdot \tfrac{|YA|}{|AB|}\right) \cdot \\ &\left(\underline{\tfrac{|XB|}{|BY|}} \cdot \tfrac{|YB|}{|BA|} \cdot \tfrac{|AB|}{|BX|}\right) \cdot \left(\underline{\tfrac{|YA|}{|AX|}} \cdot \tfrac{|XA|}{|AB|} \cdot \tfrac{|BA|}{|AY|}\right) \quad = \quad 1. \end{aligned}$$

Ratios that do not cancel are underlined. They yield exactly the desired expression. The combinatorics of this expression is again a kind of tunnel: A tetrahedron in which two holes are cut along opposite edges. If we close these two slots of the tetrahedron by two pockets, we immediately get a proof for the following incidence theorem:



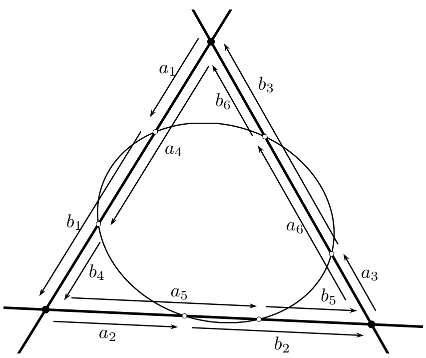




The picture on the right indicates the manifold structure with complete labels for the vertices and for the edge points. The Menelaus triangles are marked by a black dots.

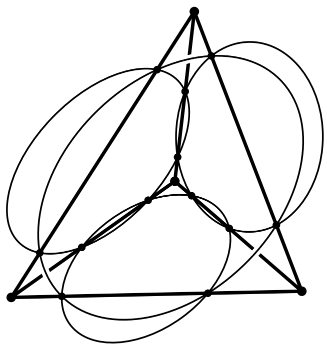
## 5 Other primitives

So far we used only Ceva configurations and Menelaus configurations as basic entities. They both "lived" on a triangle and had exactly one edge point per edge. In this section we want to give a glimpse of what happens if we allow more than one edge point or if we consider polygons other than triangles.

**5.1 Conics.** There is a beautiful theorem of Carnot, which can be considered to be a generalization of Ceva's Theorem.




Carnots Theorem



Gluing many Carnot configurations

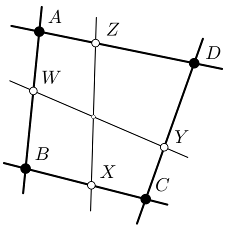
We consider a triangle with exactly two (distinct) edge points per edge. We assume the edge points are labeled $1, \ldots, 6$ and the corresponding length ratios are $a_i/b_i$. Carnot's Theorem states that we have

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \frac{a_3}{b_3} \cdot \frac{a_4}{b_4} \cdot \frac{a_5}{b_5} \cdot \frac{a_6}{b_6} \quad = \quad 1$$

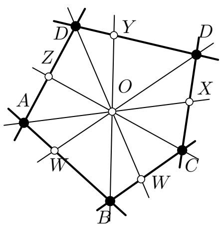
if and only if the six edge points lie on a common conic. We can immediately use Carnot's configuration as a "primitive" to build theorems that also involve conics. In the above picture on the right we present a small theorem that only involves Carnot-faces: If one has a tetrahedron with two distinct points on each edge and if the six edge points of three faces are co-conical, then they will be co-conical for

the last face automatically. In fact, there is nothing special about the tetrahedron. Any oriented triangulated 2-manifold would serve as a frame as well.
It is also easily possible to prove, for instance, Pascal's Theorem with this method. For this one combines one Carnot configuration with four Menelaus configurations. Since each edge of the Carnot configuration has two edge points, one has to count them with multiplicity two and each of these edges has to be glued to two Menelaus configurations – one for each edge point.

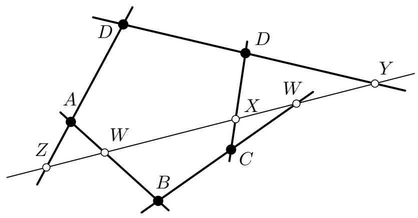
**5.2 $n$-gons.** If we consider $n$-gons instead of triangles we immediately can produce several nice generalizations of Ceva's and Menelaus's Theorem. Many of them have been studied in the literature. Most of them are of the form that a certain (combinatorially symmetric) incidence configuration forces a certain product of length ratios (often with cyclic symmetry) to be 1 or $-1$. For an extensive treatment of this topic see the article series [12, 13, 14, 15, 21]. Here we only want to present three of these theorems, without proofs.



A lifting condition



Hoehn's Theorem



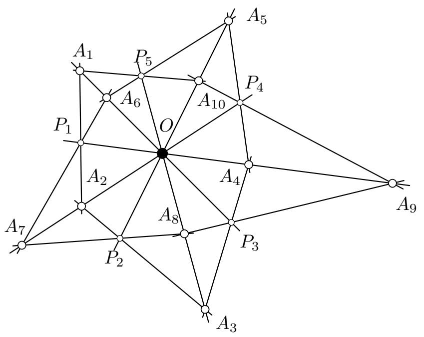
General Menelaus

If we consider a quadrangle $A, B, C, D$ (first picture above) with edge points $W$, $X$, $Y$, $Z$, then the condition

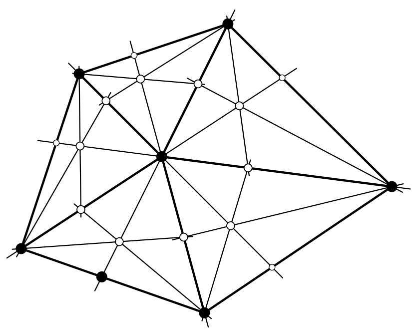
$$\frac{|AW|}{|WB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YD|} \cdot \frac{|DZ|}{|ZA|} = 1$$

holds if and only if the six lines of the picture are tangent to a common conic. Another equivalent condition for this is that there is a proper lifting of these lines to three-space that preserves all nine incidences.
The second drawing shows a theorem which is known under the name *Hoehn's Theorem*. Consider an $n$-gon with odd $n$ and an additional point $O$. If we construct edge points by intersecting each edge with the line that connects $O$ to the opposite vertex, then the cyclic product of the length ratios will be 1. This theorem is a direct generalization of Ceva's Theorem where we have $n = 3$. It has to be mentioned that the converse of Hoehn's Theorem is not true for $n > 3$, since the cyclic product being 1 does not necessarily imply that all central lines meet in a point (which can be seen by a simple degree-of-freedom count). One can prove this theorem by exactly the same idea we used in Section 2.1 to prove Ceva's Theorem.
The last picture shows a generalization of Menelaus's Theorem. Consider an $n$-gon where edge points are generated by cutting the edges with a single line (not necessarily in the interior) then the cyclic product of the length ratios equals $(-1)^n$. The theorem can be proved easily by triangulating the $n$-gon, applying the usual Menelaus Theorem to all triangles and canceling all interior edges.

As one application for using Hoehn's Theorem as a basic building block we consider a theorem of Saam, whose incidence configuration is given in the left picture below (compare [19, 20]). One starts with a central point and an odd number of lines passing through it. On each of these lines one chooses a projection point $P_i$. Then one starts with a point $A_1$ (compare to the picture) and projects this point through $P_1$ onto the next line. If one proceeds projecting, then after cycling around twice one again reaches the initial point $A_1$. The picture on the right shows a way to prove this theorem by a cycle construction. One forms an $n$-gon by the points $A_i$ with odd $i$ and forms a pyramid over this $n$-gon. On all triangular faces one imposes a Ceva configuration and on the $n$-gon one imposes a Hoehn configuration. The usual cancellation procedure proves the theorem. It also becomes obvious that Saam's theorem is only a special case of a more general theorem that has the same underlying manifold proof, but in which the apex of the pyramid does not coincide with the center-point of Hoehn's configuration.



Saam's Theorem


The underlying manifold

## 6 The connection of Ceva/Menelaus proofs and binomial proofs

In this section we will discuss the connection of Ceva/Menalaos proofs with the binomial proving techniques presented in Section 1. At first sight it is not clear at all whether and how the two proving concepts are related. In particular, binomial proofs seem to carry much less structure (they "just" admit a cancellation pattern but no obvious topological structure) than Ceva/Menelaus proofs. Therefore it would not be surprising if the class of theorems provable by binomial methods would be considerably richer than the class of theorems provable by Ceva/Menelaus constructions.
Amazingly, this is not the case. There is even an algorithmic way of systematically translating one structure into the other. Space restrictions do not allow us to give a full formal proof of this result here. We only want to sketch the basic ideas and demonstrate the relation by a concrete example.

**6.1 Symbols and relations.** One problem of the formal treatment of our proving techniques is that we treat the same object on very different levels. First of all there is the combinatorial structure of the incidence theorem that serves as a basis for all other levels. Secondly there are concrete realizations of the hypotheses and non-degeneracy requirements of the theorem, in which we have concrete coordinates and can calculate concrete values for determinants and length ratios.

On the other hand, if we consider binomial proofs, the determinants play merely the role of formal symbols. We will not assign specific values to them, but the incidence structure of the theorem determines rules how these formal symbols are related to each other. In the case of binomial proofs the decisive relations on the symbols are given by the identities of the form $[\ldots][\ldots] = [\ldots][\ldots]$ that are used in our proofs. If we consider Ceva/Menelaus proofs instead, the role of formal symbols is played by the ratios of oriented lengths. The existence of Ceva or Menelaus sub-configurations is translated into the presence of relations among these length ratios.

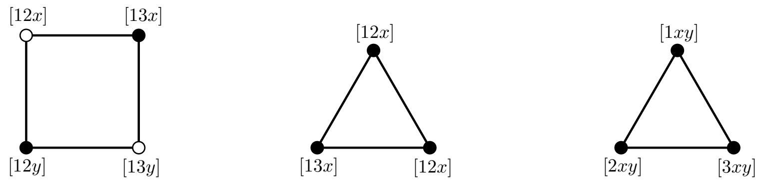
The essence of our proofs is that the purely formal treatment of the symbols (formal determinants or formal length ratios) allows to conclude additional relations that have to be present in every realization of the hypotheses configuration. The idea of treating the abstract properties of an incidence configuration on a formal level in not at all new. In his article *The bracket ring of combinatorial geometry* [23] N. White discusses extensively the ring structure of formal determinants that is imposed by an underlying incidence structure. Similarly, in a series if articles on the *Tutte Group* of a matroid [9, 10, 11, 22] A. Dress and W. Wenzel analyze the consequences of incidence relations on the abstract multiplicative group on several formal kinds of symbols. The abstract symbols that are studied there are *abstract determinants*, *ratios of (adjacent) abstract determinants*, and *abstract scalar products*. For a fixed underlying matroid each of these setups produces a group and the three groups are very closely related. Up to isomorphism they just differ by a factor of $\mathbb{Z}^k$. The isomorphism is not at all trivial and relies on homotopy arguments on so called Maurer graphs [10, 16].

It turns out that the essence of our proving techniques can be nicely expressed in terms of Tutte groups and it turns out that the equivalence of binomial proofs and Ceva/Menelaus proofs relies on the equivalence of the different setups for Tutte groups. For reasons of space limitations we will not present the formal setup here, since it involves quite a lot of technical machinery. Rather than that we will explain the basic concepts and describe the translation process by a concrete example.

**6.2 Binomial proofs vs. Ceva/Menelaus proofs.** For our explanations we will make a few simplifying assumptions. We will assume that we study a concrete incidence configuration with a concrete matroid underlying it. We restrict the considerations to the rank 3 case only. The hypotheses (and the conclusion) of our theorem will be expressed by certain non-bases of the matroid. The non-degeneracies will result in the presence of certain bases of the matroid. In general an incidence theorem will not determine the underlying matroid uniquely. Only a partial structure of it will be fixed by the hypotheses non-degeneracies. We will neglect this technical difficulty and assume that we have a fixed underlying matroid $\mathbf{M}$ with set of bases $\mathbf{B}$.

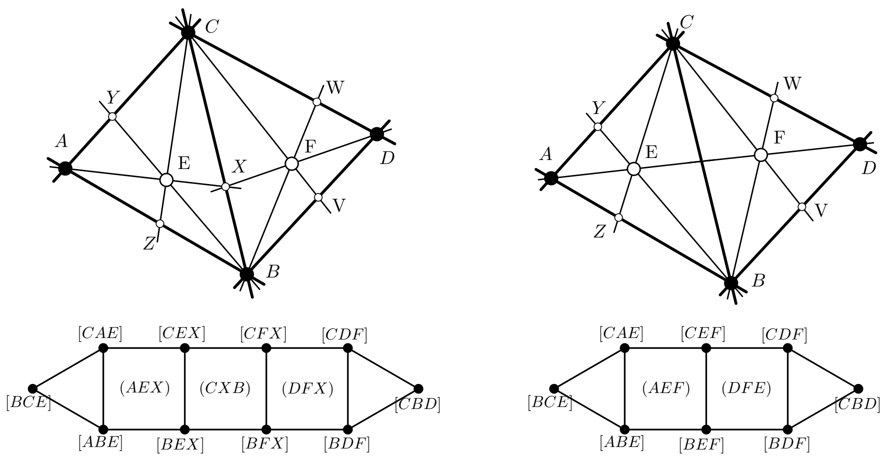
We consider the graph $\mathcal{G}_{\mathbf{M}}$ whose vertices correspond to the bases $\mathbf{B}$ and whose edges are those pairs of bases that differ exactly by one element. We will identify each determinant that occurs in a binomial proof (and therefore will be a basis) with the corresponding vertex of $\mathcal{G}_{\mathbf{M}}$. Assume that $(1,2,3)$ are collinear and that $[1,2,x][1,3,y] = [1,2,y][1,3,x]$ is a corresponding binomial equation. Bases involved in such a binomial equation correspond to a 4-cycle in $\mathcal{G}_{\mathbf{M}}$ without diagonal edges. We will call such a quadrangle a *binomial quadrangle*. A binomial quadrangle with vertices $(1,2,x),(1,3,y),(1,2,y),(1,3,x)$ is called *degenerate* if at least

one of the triples $(1,2,3)$ or $(1,x,y)$ is not a basis. If both triples are non-bases it is called *pure*.
We will bicolor degenerate quadrangles, which correspond to binomial equations, such that the bases on the left of the equations are colored white and the bases on the right are colored black. A binomial proof now corresponds to a collection of (bicolored) binomial degenerate quadrangles, such that in this collection each vertex is as often colored black, as it is colored white.




a binomial quadrangle a triangle of first kind a triangle of second kind

In the picture above the first drawing shows such a quadrangle with labeled bases for the binomial relation $[1,2,x][1,3,y] = [1,2,y][1,3,x]$. In the graph $\mathcal{G}_{\mathbf{M}}$ there can be exactly two combinatorially different types of triangles. They are represented by the two labeled triangles in the picture above. In the context of Tutte groups these triangles are called *triangle of first kind* and *triangle of second type*, respectively. One of the fundamental properties of such bases graphs of matroids is that every cycle in the graph can be generated as a sum of triangles and of pure quadrangles (here sum is meant in the sense that if coinciding edges are added with opposite directions, then they cancel each other). If the rank three matroid does not contain loops or parallel elements, each cycle can even be considered as sum of triangles only. This result is the technical kernel of the fact that the different Tutte groups are isomorphic up to a $\mathbb{Z}^k$-factor. It will also be our main tool for the translation of binomial proofs into Ceva/Menelaus proofs.

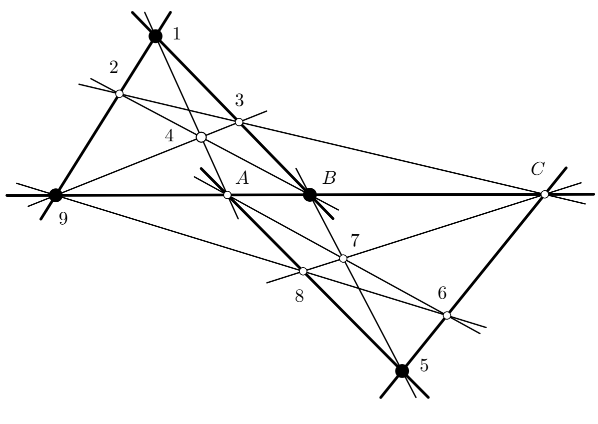



Bases graphs of glued Ceva configurations.

There is a close connection of triangles of first and second kind to Ceva's and Menelaus's configuration. For this we consider the bases as non-zero triangle areas in our configuration. An oriented edge can be interpreted as a quotient of two such triangle areas. As explained in Section 2.1. such a quotient can also be considered as a ratio of lengths on a line. The three ratios encoded by each of the two triangles are exactly those ratios that we used in our original proofs of Ceva's and Menelaus's Theorem in Section 2.1.

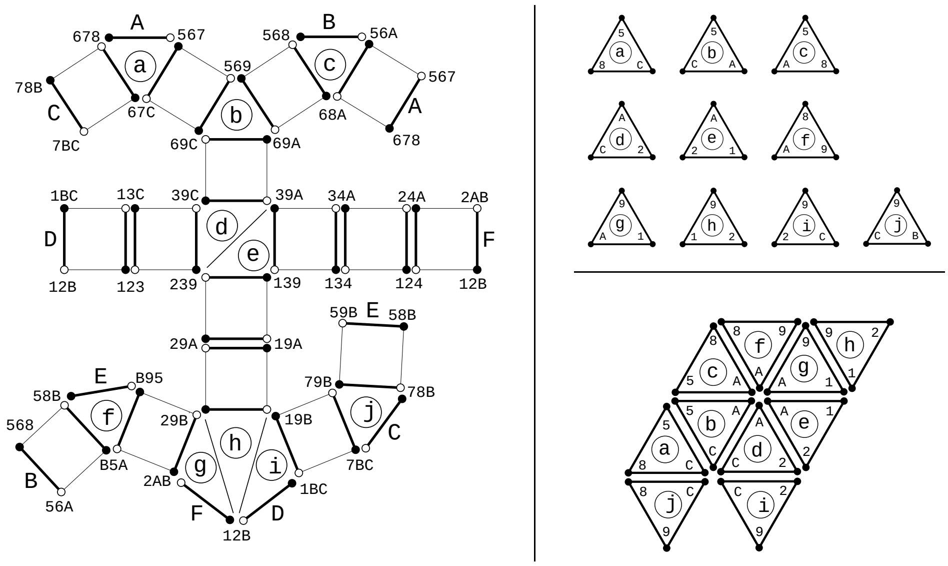
Now, let us study the situation where in a Ceva/Menelaus proof two triangles are glued along an edge. What does the corresponding situation in the bases graphs look like? We will only elaborate on two situations; the rest is essentially analogous. Let us consider the situations shown in the picture on the preceeding page, where two Ceva configurations are joined. In each of the corresponding cases a substructure of the basis graph is shown. The quadrangles that are visible in the graphs are actually degenerate binomial quadrangles (the triple inside the quadrangle indicates the non-basis that is responsible for the degeneracy). Recall that the quadrangles represent a relation of the form $\frac{[1,a,x]}{[1,a,y]} = \frac{[1,b,x]}{[1,b,y]}$. Both sides of the equation can be interpreted as representing the same length ratio, however represented by different area quotients. The chains of quadrangles that occurs in the examples above serve as a kind of translator between the two Ceva triangles. They make sure that the area ratios in one triangle really represent the same length ratio as the area ratio of the other triangle. If we consider the bases graph underlying a configuration that admits a Ceva/Menalaus proof, we will find the following substructure. Each Ceva or Menelaus configuration of the proof corresponds to a suitable triangle in the graph. All edges of these triangles are paired by chains of degenerate binomial quadrangles (this chain may have length zero). The collection of binomial equations corresponding to exactly these quadrangles forms a binomial proof of the theorem. This gives the translation of Ceva/Menelaus proofs into binomial proofs.

Slightly more complicated is the translation of a binomial proof into a Ceva/Menelaus proof. (this is not very surprising, since the second kind of proof carries much more structural information). We will demonstrate the basic procedure by an example that shows essentially all important features. We revisit the theorem below (recall Section 4.3) and consider a concrete proof that was generated by an implementation of an algorithm that produces binomial proofs.




```
[239][19A] =  [139][29A]
[134][24A] =  [124][34A]
[124][2AB] = -[12B][24A]
[12B][13C] =  [123][1BC]
[123][39C] = -[239][13C]
[139][34A] =  [134][39A]
[29A][19B] =  [19A][29B]
[29B][5AB] =  [2AB][59B]
[568][69A] =  [569][68A]
[67C][569] =  [567][69C]
[56A][58B] =  [568][5AB]
[678][7BC] = -[67C][78B]
[567][68A] = -[56A][678]
[39A][69C] =  [39C][69A]
[1BC][79B] =  [19B][7BC]
[58B][79B] =  [78B][59B]
```

How can we derive a Ceva/Menelaus proof from such a given binomial proof? How can we reveal and reconstruct the manifold structure? Here we will give a kind of cooking recipe without a formal explanation why this recipe works. As a first step we reconstruct a suitable substructure of the underlying basis graph. For each binomial equation of the proof we form a quadrangle with bicolored vertices and label the vertices by the corresponding bases. The bases on the left sides of the binomial equations will be colored white, the others will be colored black. We then try to stick the quadrangles together such that each white vertex meets its black counterpart (see picture below – capital letters on an exterior edge mean that this edge has to be identified with the corresponding edge labeled by the identical letter). This will always be possible since it was the defining property of a binomial proof. We will now interpret the quadrangles as substructures that are used to link Ceva/Menelaus configurations. For each quadrangle we have to make a decision which pair of opposite sides is considered as carrying the information of the length ratio. In our picture we have drawn these edges slightly darkened (observe that there is a lot of freedom in this choice). If two quadrangles are attached to each other, such that the two dark edges coincide we can neglect these two edges for the further considerations. We can also neglect the non-darkened edges. We are left with a collection of darkened edges that will form several edge-disjoint cycles. In our example we have seven such cycles: five triangles, one quadrangle and a pentagon. The choice which pair of edges will be darkened has a great influence on the number and size of cycles we will get. In our example we made this choice in a way that a maximum number of dark edges became identified and such that we get many small cycles.



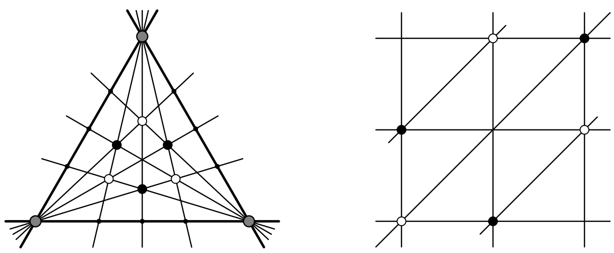

Now, Maurer's homotopy theorem tells us, that each cycle can be decomposed into triangles (perhaps for this we have to use new bases that are not already present in the substructure constructed so far. In our example this is not the case). In our example the quadrangle can be decomposed into two triangles and the pentagon is decomposed into three triangles. The collection of all these triangles forms the support for all our Ceva/Menelaus configurations that are needed for a Ceva/Menelaus proof. From the bases labels at the vertices of the graph we can easily read off whether a triangle has to be a considered as a Ceva or as a

Menelaus configuration. In our example only the triangles labeled “c” and “h” carry Ceva configurations (in fact they are toothpicks). All other triangles are Menelaus configurations. We can also easily read off the vertices for each triangle. Finally, we can take all these triangles with vertices labelled by the corresponding points and can construct a closed and oriented manifold from them. In fact, the basic gluing structure is already given by the chaining quadrangles. However, we have to be a little careful to assign the correct orientation to the Ceva configurations (in the Ceva/Menelaus manifold this will be opposite to the picture of the quadrangle cancellations.). In our example the resulting manifold (right lower picture) turns out to be a sphere. This finishes the process of translating a binomial proof into a Ceva/Menelaus proof.

## 7 Spaces of theorems

In this section we want to give a glimpse on another aspect of Ceva/Menelaus proofs — namely how different proofs can be combined to form new proofs of more complicated theorems. We will study this in the simplest possible case, for which indeed even a complete classification is possible.

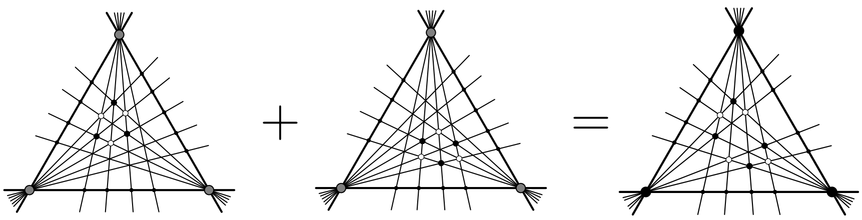
**7.1 Grid theorems.** For this we now come back to a structure that we have already met in Section 3.4, when we proved Pappos’s Theorem. We will study those incidence theorems for which the Ceva/Menelaus proof requires only Ceva triangles and for which furthermore all Ceva triangles have the same vertices $1, 2, 3$. Pappos’s Theorem was the smallest such example. In such a theorem we have three bundles of lines each bundle passing through one of the three vertices. Each Ceva point is met by one line of each bundle. Since the position of each line is uniquely determined by the length ratio of the corresponding Ceva point, there must be a manifold proof if the underlying incidence structure is a theorem. If we have a manifold proof for such a theorem, then adjacent triangles of the manifold must have opposite orientation with respect to the triangle $(1, 2, 3)$. In our configuration we color the Ceva points that use one orientation white and the others black. The crucial condition for us to have a theorem is that along each line we have as many black points as we have white points. The picture below shows the situation for Pappos’s Theorem. The drawing in the middle is the situation in which the vertices of the triangle were moved to the line at infinity. We can associate a matrix to the situation that has an entry “+1” for every white point, a “-1” for every black point and zero otherwise. Our theorem property now translates into the fact, that such a matrix has columns sums, row sums and sums in the north-east diagonal direction all zero.



$$\begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{pmatrix}$$

Pappos’s Theorem and its underlying matrix

**7.2 Spaces of incidence theorems.** Now, conversely consider an $n \times n$ matrix with integer entries and column sums, row sums, north-east diagonal sums being zero. Without loss of generality one can assume that at least one entry of the matrix is odd. If not, divide the matrix by a suitable power of two. The non-zero entry of the matrix will be called its *support*. Take a triangle $(1, 2, 3)$ as frame for an incidence configuration. Each row that has a support entry will correspond to a Ceva line through point 1, each column that contains a support entry will correspond to a Ceva line through point 2 and similarly each diagonal to a Ceva point through point 3. The support entries of the matrix correspond to the Ceva points, in which the corresponding row, column and diagonal lines meet. By this construction each support entry corresponds to a Ceva triangle that has to be counted with a multiplicity induced by the value of the matrix entry (the sign indicates the orientation). The cancellation pattern on the manifold is directly induced by the sum$= 0$ property of the matrix. We can conclude that the Ceva configuration at the odd support entry with value $s$ must satisfy a relation $\left(\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|}\right)^s = 1$ and hence (at least over the real numbers) it is the conclusion of an incidence theorem. Observe that the column sum, row sum, diagonal sum conditions are just linear conditions over the vectorspace of $n \times n$ matrices. A careful count shows that exactly $4n - 4$ of them are linearly independent. Thus the space of all admissible matrices is generated by a suitable basis of $(n - 2)^2$ configurations (which have to be incidence theorems as well). It turns out that a nice basis is given by the Pappos configurations that are supported by adjacent rows and columns (there are exactly $(n - 2)^2$ of them). The picture below shows an example in which an incidence theorem on a $4 \times 4$ grid is generated as the sum of two Pappos configurations. Topologically this process of forming a sum corresponds to a surgery on the two Pappos tori. The tori are identified along two triangles, and after this the triangles are removed from the manifold.

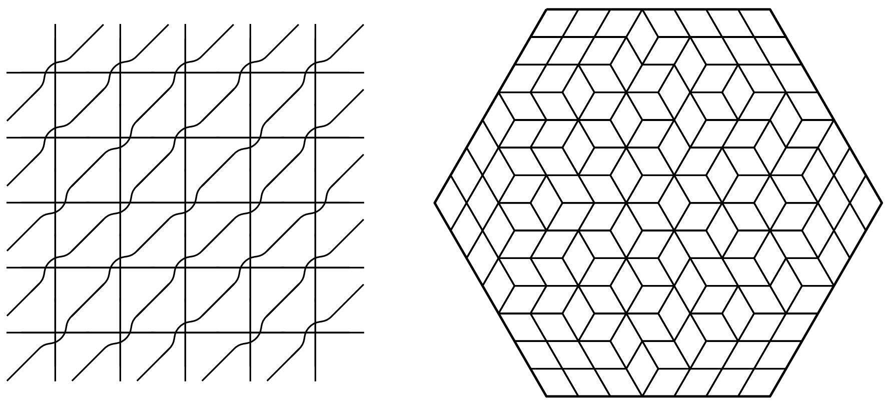


## 8 Conclusion

We just scratched the surface. In a sense this article is the outline of a whole program: How can cycle structures on manifolds be used to get a deeper understanding of theorems in geometry? There are many things that still have to be investigated. We want to mention a few of them here.

- *What happens if we allow fields other than $\mathbb{R}$?* Over the complex numbers the simple distinction of $+1$ and $-1$ evolves to an entire spectrum of possible units — new aspects of torsion enter the theory. Already simpler configurations result in non-trivial binomial relations. For instance the fact that $A, B, C, D$ are cocircular in $\mathbb{CP}^1$ results in the fact that the cross ratio of these points is real. It is an easy task to take such cross ratio quadrangles and form manifolds from them that prove incidence theorems (see for instance [2, 21]).

- *What about second and third order syzygies?* In this article we studied relations among sub-configurations of an incidence theorem. In the last section we saw that there are cases in which spaces of such theorems emerge. What can we say about relations between these relations, or even about relations on the relations between the relations. At least in the context of the bracket ring such problems have been partially studied [7].
- *Back to non-realizability proofs.* The origin of the whole investigations were non-realizability proofs for arrangements of pseudolines. How can the results in this paper, be applied to get classification results with respect to realizability and non-realizability in this context. In particular, realizability of pseudoline arrangements that consist of three bundles of pseudolines can be characterized completely by the methods in Section 7. This implies also a characterization of liftable rhombic tilings with three directions.
- *Where is the complexity?* Proving theorems is provably hard. So the question arises about the limits of the proving techniques described here. What do theorems look like, that cannot be proved by Ceva/Menelaus proofs? Can the theory be extended to cover even these theorems?
- *Relations to integrability theory.* There is a whole community that works on a setup for a theory of discrete analogues for integrable and differentiable structures (see for instance [1, 3, 4]). In these setups smooth structures are replaced by discrete samples of points that still carry fundamental properties of integrability. One of these fundamental properties is that a homology theoretical discrete generalization of Green's and Stokes's Theorems hold. The structures that appear there are very similar to the cancellation patterns on manifolds. What exactly is the relation?
- *Make good implementations.* Finally, one should take advantage of the knowledge of underlying topological structures to implement automatic geometry provers that do not "blindly" test all cancellation patterns. The topological information should be used to rule out at least the most stupid dead ends of the search tree.



A perturbed arrangement of pseudolines, and a rhombic tiling

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