TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 13 (January 20, 2016) —

Problem 1. Dual lattices.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice with lattice basis $\mathcal{B} = \{b_1, \dots, b_d\} \subset \mathbb{R}^d$. Set $\widetilde{\mathcal{B}} := \{\beta_1, \dots, \beta_d\} \subset \mathbb{R}^d$ with $\langle \beta_i, b_j \rangle = \delta_{ij}$.

i) Show that $\mathsf{span}_{\mathbb{Z}}(\beta_1, \dots, \beta_d) = \Lambda^*$.

ii) Show that $\det(\Lambda^*) = \frac{1}{\det(\Lambda)}$.

iii) Describe the dual basis of Λ if $\mathsf{rk}(\Lambda) < d$ and prove $\det(\Lambda^*) = \frac{1}{\det(\Lambda)}$ in this case.

Problem 2. Width of an ellipse.
Construct a convex body K of lattice width 1 in the 2-dimensional standard lattice $\Lambda = \mathbb{Z}^2$ such that $\omega(K, v) \neq 1$ for all shortest vectors $v \in \mathbb{Z}^2$.

Problem 3. Upper bound for $\mu(\Lambda)\rho(\Lambda^*)$.
Prove the missing step for the inductive proof of $\mu(\Lambda)\rho(\Lambda^*) \leq \frac{1}{4}\sqrt{\sum_{k=1}^{d} k^2}$.

Problem 4. Tangent line of an ellipse.
Consider the ellipse $E := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{(x-\tau)^2}{\alpha^2} + \frac{(y-\sigma)^2}{\beta^2} \leq 1 \right\}$ and $z = (z_1, z_2) \in \partial E$.
Show that the line through z that is tangent to ∂E satisfies $\frac{(z_1-\tau)(x-\tau)}{\alpha^2} + \frac{(z_2-\sigma)(y-\sigma)}{\beta^2} = 1$.
Hint: Prove the claim first for $\tau = \sigma = 0$.