TECHNISCHE UNIVERSITÄT MÜNCHEN

Zentrum Mathematik

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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 11 (December 23, 2015) —

Problem 1. Continued fractions.

The continued fraction expansion $[a_0; a_1, \dots, a_n, \dots]$ of $a \in \mathbb{R}$ is computed by the following iteration:

i) Set $i := 0$, write $a = \lfloor a \rfloor + \{a\}$ and set $a_i := \lfloor a \rfloor$.
ii) If $\{a\} = 0$ then stop.
iii) Set $b = \frac{1}{\{a\}}$, update i by $i + 1$, write $b = \lfloor b \rfloor + \{b\}$ and set $a_i := \lfloor b \rfloor$.
iv) If $\{b\} = 0$ then stop.
v) Update b by $\frac{1}{\{b\}}$ and i by $i + 1$, coninue at step iii).

Then $a = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_n}}}$ if the procedure stops after n iterations or $a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}$ otherwise.

a) Compute the continued fraction expansion of $\frac{164}{31}$.
b) Show that the continued fraction expansion of $a \in \mathbb{R}$ is finite if and only if a is rational.

Problem 2. i^{th} convergents for $a = [a_0; a_1, \dots, a_n] \in \mathbb{Q}$.

Let $a = [a_0; a_1, \dots, a_n]$ be a continued fraction expansion of $a \in \mathbb{Q}$ with $n \geq 2$. For $0 \leq i \leq n$, define the i^{th} convergent of a as $\frac{p_i}{q_i} = [a_0; a_1, \dots, a_i]$ with $p_i, q_i \in \mathbb{Z}$.

Prove the following identities:

a) $p_i = a_i p_{i-1} + p_{i-2}$ for $2 \leq i \leq n$.
b) $q_i = a_i q_{i-1} + q_{i-2}$ for $2 \leq i \leq n$.
c) $p_{i-1} q_i - p_i q_{i-1} = (-1)^i$ for $1 \leq i \leq n$.

Problem 3. Continued fractions and cones.

For $p, q \in \mathbb{N}$ coprime let $[a_0; a_1, \dots, a_n]$ be the continued fraction expansion of $\frac{p}{q}$ and $\frac{p_i}{q_i} = [a_0; a_1, \dots, a_i]$ be the i^{th} convergents for $i = 0, 1, \dots, n$. Moreover, consider the following cones:

$$K := \operatorname{cone}\left\{\binom{1}{0}, \binom{q}{p}\right\}, \qquad K_{-1} := \operatorname{cone}\left\{\binom{1}{0}, \binom{0}{1}\right\},$$
$$K_0 := \operatorname{cone}\left\{\binom{0}{1}, \binom{1}{p_0}\right\}, \text{ and} \qquad K_i := \operatorname{cone}\left\{\binom{q_{i-1}}{p_{i-1}}, \binom{q_i}{p_i}\right\} \text{ for } i \in [n].$$

Moreover, let $[A] : \mathbb{R}^2 \to \{0, 1\}$ defined via $[A](x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ be the characterstic function of $A \subset \mathbb{R}^2$.

a) Prove that the cones $K_{-1}, K_0, \dots, K_n$ are unimodular.
b) Prove that $[K] = \sum_{i=-1}^{n} (-1)^{i+1} [K_i]$ if n is odd.
c) Prove that $[K] = [R] + \sum_{i=-1}^{n} (-1)^{i+1} [K_i]$ if n is even where R is the ray emanating from the origin in the direction of (q_n, p_n).
d) Illustrate the the cutting and pasting of unimodular cones associated to the continued fraction expansion of $\frac{164}{31}$.