TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 10 (December 16, 2015) —

Problem 1. Lattice determinant and basis vectors.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice with basis $\mathcal{B} = \{b_1, \dots, b_d\}$ and associated Gram-Schmidt-orthogonalized basis $\{w_1, \dots, w_d\}$. Show that $\det(\Lambda) = \prod_{i\in[d]} \|w_i\|$.

Problem 2. Volume of a euclidean d-ball.
Consider the d-dimensional euclidean space $\mathbb{R}^d$ that is endowed with the standard euclidean norm $\|\cdot\|$, the euclidean ball $B_r^d := \left\{x \in \mathbb{R}^d \;\middle|\; \|x\| \le r\right\}$ of radius $r > 0$ centred at 0 and the Gamma function $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ defined via $x \mapsto \int_0^\infty t^{x-1}e^{-t}dt$. Show that

$$\mathrm{vol}_{\mathbb{R}^d}(B_r^d) = \frac{\pi^{\frac{d}{2}} r^d}{\Gamma\left(\frac{d}{2}+1\right)}.$$

Problem 3. Packing radius of a lattice.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice with lattice basis $\mathcal{B} = \{b_1, \dots b_d\}$. The packing radius $\rho(\Lambda)$ of Λ is the largest $\rho > 0$ such that the open d-balls $\mathrm{int}(B_\rho^d(x))$ and $\mathrm{int}(B_\rho^d(y))$ of radius ρ centred at x, y do not intersect for any distinct pair of point $x, y \in \Lambda$. Then

$$\rho(\Lambda) \le \frac{\sqrt[d]{\det(\Lambda)\cdot\Gamma\left(\frac{d}{2}+1\right)}}{\sqrt{\pi}}.$$

Problem 4. Upper bound for smallest non-trivial lattice point.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice with lattice basis $\mathcal{B} = \{b_1, \dots b_d\}$ and $x \in \Lambda \setminus \{0\}$. Then $\|x\| \le \sqrt{d}\cdot\sqrt[d]{\det(\Lambda)}$.

You may use (and prove) the following useful identities of the Gamma function:
i) $\Gamma(x+1) = x\Gamma(x)$ for all $x \in \mathbb{R}^+$
ii) $\Gamma(x) = (x-1)!$ for all $x \in \mathbb{N}$
iii) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$