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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 9 (December 09, 2015) —

Problem 1. Polytopes are shellable.
A *shelling* of a pure k-dimensional polytopal complex $\mathcal{C}$ is a linear ordering F_1, F_2, …, F_s of its facets such that either $\mathcal{C}$ is 0-dimensional, that is, the facets are points, or it satisfies the following conditions:

i) The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet has a shelling.

ii) For $1 < j \leq s$, the intersection of the facet F_j with the previous facets is nonempty and is a beginning segment of a shelling of the $(k-1)$-dimensional boundary complex ∂F_j of F_j, that is,

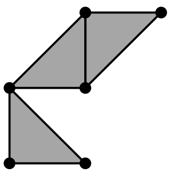
$$F_j \cap \left(\bigcup_{i\in[j-1]} F_i\right) = G_1 \cup G_2 \cup \cdots \cup G_r$$

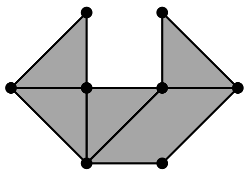
for some shelling G_1, G_2, …, G_t of $\mathcal{C}(\partial F_j)$ and $r \in [t]$.

Answer the following questions about shellings of 1- and 2-dimensional complexes.

a) Prove that a 1-dimensional polyhedral complex $\mathcal{C}$ is shellable if and only if $\mathcal{C}$ is a connected graph.

b) Prove or disprove the existence of a shelling for the following polyhedral complexes.

a) 

b) 

Problem 2. Polytopes are shellable.
Prove the following statements.

a) If F_1, F_2, …, F_s is a shelling for the boundary ∂P of a polytope P then F_s, F_{s-1}, …, F_1 is also a shelling for ∂P.

b) Let $P \subset \mathbb{R}^d$ be a d-polytope and choose $x \in \mathbb{R}^d \setminus P$ such that x is not contained in the affine hull of any facet F of P. If $\mathcal{F} := \{F_1, \ldots, F_s\}$ is the inclusion-maximal set of facets of P such that x is beyond each $F \in \mathcal{F}$ then ∂P has a shelling in which the facets of $\mathcal{F}$ come first.
Hint: Construct a shelling using a suitable line through x and $y \in \operatorname{int}(P)$.

Problem 3. Euler-Poincaré formula for polytopes.
Prove the Euler-Poincaré-formula for a d-polytope P with f-vecrtor $f(P) := (f_0, f_1, \ldots, f_d) \in \mathbb{N}^d$:

$$\sum_{i=0}^{d} (-1)^i f_i = 1.$$

Hint: Extend the notion of an f-vector to a polyhedral complex $\mathcal{C}$ of dimension at most d and consider the (reduced) Euler characteristic $\chi(\mathcal{C}) := (-1) + \sum_{i=0}^{d} f_i$. Show that χ is *additive*, that is, χ satisfies $\chi(\mathcal{D}) + \chi(\mathcal{D}') = \chi(\mathcal{D} \cap \mathcal{D}') + \chi(\mathcal{D} \cup \mathcal{D}')$ for any pair $\mathcal{D}$, $\mathcal{D}'$ of polyhedral complexes where $\mathcal{D} \cup \mathcal{D}'$ is again a polyhedral complex. Then use Problem 2 to prove $\chi(\mathcal{P}) = 0$ for every polytope P by induction and conclude the Euler-Poincaré formula.

Problem 4. Ehrhart polynomial via Brion's Theorem.
Let $C_2 = \mathsf{conv}\{0, e_1, e_2, e_1 + e_2\}$ be the 2-dimensional standard cube. Use Brion's Theorem to show that $\mathsf{ehr}_{C_2}(k) = 1 + 2k + k^2$.