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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 8 (December 02, 2015) —

Problem 1. Coefficients of Ehrhart polynomials are rational.
Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with vertex set $\mathsf{ver}(P) = \{v_1, \ldots, v_2\} \subset \mathbb{Z}^d$ and Ehrhart polynomial $\mathsf{Ehr}_P(t) = \sum_{i=0}^{d} c_i t^i$. Prove that $d! c_i \in \mathbb{Z}$ for all $i \in [d]$.

Problem 2. More on generating functions.
Let $f \in \mathbb{R}[t]$ be a polynomial of degree $\deg(f) = d$ such that $\sum_{t \in \mathbb{N}_0} f(t) z^t = \frac{h_0^* + \ldots + h_d^* z^d}{(1-z)^{d+1}}$. Prove that $h_{k+1}^* = h_{k+2}^* = \ldots = h_d^* = 0$ and $h_k^* \neq 0$ if and only if $f(-1) = f(-2) = \ldots = f(-(d-k)) = 0$ and $f(-(d-k+1)) \neq 0$.

Problem 3. The leading coefficient of the h^*-polynomial and lattice points of $\operatorname{int}(P)$.
Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope and $h^*(t)$ its h^*-polynomial and $\mathsf{ehr}_P(t)$ its Ehrhart polynomial. Then $h^*_{\deg(P)} = \mathsf{ehr}_P(-d + \deg(P) - 1)$ and the coefficient h_d^* of $h^*(t)$ equals the the number of lattice contained in $\operatorname{int}(P)$, that is, $h_d^* = \left|\operatorname{int}(P) \cap \mathbb{Z}^d\right|$.

Problem 4. Special lattice tetrahedra.
For numbers $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$ consider the tetrahedron $\Delta_{p,q} := \mathsf{conv}\left\{0, e_1, e_2, \begin{pmatrix} 1 \\ p \\ q \end{pmatrix}\right\} \subset \mathbb{R}^3$.

a) Show that $\Delta_{p,q} \cap \mathbb{Z}^3 = \left\{0, e_1, e_2, \begin{pmatrix} 1 \\ p \\ q \end{pmatrix}\right\}$.
b) Compute the Ehrhart polynomial and the h^*-polynomial of $\Delta_{p,q}$.