TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

PD Dr. Carsten Lange

Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 7 (November 25, 2015) —

Problem 1. The Ehrhart series of the standard simplex Δ_d.
Compute the rational function associated to the Erhart series of the standard simplex Δ_d.

Problem 2. The Ehrhart series of pyramids.
Let $P \subset \mathbb{R}^d$ be a lattice polytope with vertex set $\mathsf{ver}(P) = \{v_1, \dots, v_r\} \subset \mathbb{Z}^d$ and associated Ehrhart series $\mathsf{Ehr}_P(t) = \sum_{k \in \mathbb{N}_0} \mathsf{ehr}_P(k) t^k$. Consider the pyramid $\mathsf{pyr}(P)$ over P:

$$\mathsf{pyr}(P) := \mathsf{conv}\{ \binom{v_1}{0}, \dots, \binom{v_r}{0}, \binom{0}{1} \}.$$

a) Compute the Ehrhart counting function $\mathsf{ehr}_{\mathsf{pyr}(P)}(k)$ of the pyramid over P.
b) Compute associated rational functions $\mathsf{Ehr}_{\mathsf{pyr}(P)}(t)$ of the pyramid over P.

Problem 3. Binomials as basis for polynomials.
As the classical binomial coefficient $\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{d!}$ for $n, m \in \mathbb{N}$ extends naturally to $n \in \mathbb{C}$, we define the polynomial $f_j(t) := \binom{t+d-j}{d}$ for fixed $d \in \mathbb{N}_0$ and all $j \in \mathbb{N}_0$ with $j \leq d$.

a) Prove that $\binom{t+d-1-k}{d} = (-1)^d \binom{k-t}{d}$ for all $t, k \in \mathbb{C}$ and $d \in \mathbb{N}_0$.
b) Compute the leading coefficient and the degree for each polynomial $f_j(t)$ and show that $f_0(t), \dots, f_d(t)$ is a basis for the vector space of all polynomials of degree $\leq d$.

Problem 4. Generating functions and polynomials.
Prove that $f : \mathbb{N}_0 \to \mathbb{R}$ extends to a polynomial $f : \mathbb{R} \to \mathbb{R}$ of degree d if $g : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree at most d with non-vanishing constant coefficient and $\sum_{i \in \mathbb{N}_0} f(i) z^i = \frac{g(z)}{(1-z)^{d+1}}$.