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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 6 (November 18, 2015) —

Problem 1. The near and far half-open cones and parallelepipeds.
Let $\mathcal{B} = \{v_1, \dots, v_d\} \subset \mathbb{R}^d$ be linearly independent and $x \in \mathbb{R}^d$ be generic with respect to $C_{\mathcal{B}} := \mathsf{cone}(\mathcal{B})$.

a) Determine the number of distinct near half-open cones $C_{\mathcal{B}}[x)$ and near half-open parallelepipeds $\mathsf{epi}_{\mathcal{B}}[x)$ and draw the possible configurations for $d = 2$.
b) Is every near half-open cone also a far half-open cone? Is the same statement true for near and far half-open parallelepipeds?
c) Can a near half-open cone $C_{\mathcal{B}}[x)$ (or a near half-open parallelepiped $\mathsf{epi}_{\mathcal{B}}[x)$) be open in $\mathbb{R}^d$? Can it be closed?

Problem 2. Triangulations.
Consider polyhedra $P_{\mathcal{B}} = \mathsf{conv}(\mathcal{B})$ and $C_{\mathcal{B}} = \mathsf{cone}(\mathcal{B}')$ with $\mathcal{B} = \left\{ \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right) \right\}$ and $\mathcal{B}' = \left\{ \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\-1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\-1\\1\end{smallmatrix}\right) \right\}$.

a) Describe a triangulation $\mathcal{T}$ of $P_{\mathcal{B}}$ that does not use new vertices and a triangulation $\widetilde{\mathcal{T}}$ that uses all lattice points $P \cap \mathbb{Z}^2$.
b) Embed $P_{\mathcal{B}} \subset \mathbb{R}^2$ in $\mathbb{R}^3$ via $x \mapsto \left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)$ and describe two triangluations $\mathcal{T}'$ and $\widetilde{\mathcal{T}}'$ of $C_{\mathcal{B}'} := \mathsf{cone}(\mathcal{B}')$ that restrict to your triangulations $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ of P.
c) Partition $C_{\mathcal{B}}$ into pairwise disjoint near half-open cones that are subcones of $\widetilde{\mathcal{T}}'$.

Problem 3. Summable Laurent series.
Recall that the Laurent polynomials $\mathsf{L} := \mathbb{R}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ form a ring and that the set of formal Laurent series $\widehat{\mathsf{L}} := \left\{ \sum_{i \in \mathbb{Z}^d} \lambda_i t^i \;\middle|\; \lambda_i \in \mathbb{R} \text{ for all } i \in \mathbb{Z} \right\}$ forms an L-module.

a) Show that the set of all summable Laurent series $\hat{\mathsf{L}}^{sum}$ forms a submodule of $\widehat{\mathsf{L}}$, that is, prove first that $\hat{\mathsf{L}}^{sum}$ is a subgroup of $(\widehat{\mathsf{L}}, +)$ and second that the scalar multiplication $p \cdot \widehat{G}$ of any Laurent polynomial $p \in \mathsf{L}$ with any summable Laurent series $\widehat{G} \in \widehat{\mathsf{L}}^{sum}$ yields an element of $\widehat{\mathsf{L}}^{sum}$.

Problem 4. Summable Laurent series and rational functions.
The field of all rational functions $\mathbb{R}(t_1, \dots, t_d)$ consists of equivalence classes $[\frac{p}{q}]$ where $p, q \in \mathbb{R}[t_1, \dots, t_d]$ and $q \neq 0$ such that $f = [\frac{p}{q}] = \left[\frac{r}{s}\right]$ if and only if $p \cdot s = r \cdot q$. The rational functions are an L-module. Define $\Phi : \widehat{\mathsf{L}}^{sum} \to \mathbb{R}(t_1, \dots, t_d)$ via $\widehat{G} \mapsto [\frac{p}{q}]$ where $p, q \in \mathsf{L}$ with $q \neq 0$ such that $q\widehat{G} = p$. Prove that Φ well-defined and that ϕ is a homomorphism of L-modules