TECHNISCHE UNIVERSITÄT MÜNCHEN
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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 3 (October 28, 2015) —

Problem 1. Another characterization of a lattice basis.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice and $\mathcal{S} = \{u_1, \dots, u_d\} \subset \Lambda$ a set of cardinality d such that the d-dimensional volume of $\overline{\Pi}_{\mathcal{S}} = \left\{ \sum_{i\in[d]} \lambda_i u_i \;\middle|\; 0 \leq \lambda_i \leq 1 \text{ for all } i \in [d] \right\}$ is equal to $\det(\Lambda) = \mathsf{vol}_{\mathbb{R}^d}(\Pi_{\mathcal{B}})$ for some lattice basis $\mathcal{B}$. Prove that $\mathcal{S}$ is a lattice basis of Λ.

Problem 2. Lattice points in a fundamental parallelepiped of a sublattice.
Let $\Lambda \subset \mathbb{R}^2$ be the lattice generated by $\mathcal{B} = \{\binom{3{,}5}{4}, \binom{1}{2}\}$ and $\Gamma \subset \Lambda$ be the sublattice generated by $\mathcal{B}' = \{\binom{10}{14}, \binom{11{,}5}{14}\}$. Determine $|\Lambda \cap \Pi_{\mathcal{B}'}|$, that is, determine the number of lattice points $v \in \Lambda$ that are contained in the fundamental parallelepiped $\Pi_{\mathcal{B}'}$.

Problem 3. Lattice bases and sublattices.
Consider the lattices Γ and Λ where Λ is generated by $\mathcal{B}_\Lambda = \{v_1, v_2\} = \left\{\binom{1}{2}, \binom{1}{0}\right\}$ while Γ is generated by $\mathcal{B}_\Gamma = \{u_1, u_2\} = \{2v_1, 3v_2\} = \left\{\binom{2}{4}\binom{3}{0}\right\}$. Clearly, Γ is a sublattice of Λ.

Construct lattice bases $\{\widetilde{v}_1, \widetilde{v}_2\}$ of Λ and $\{\widetilde{u}_1, \widetilde{u}_2\}$ of Γ such that $\widetilde{u}_1 = \alpha_1 \widetilde{v}_1$ and $\widetilde{u}_2 = \alpha_2 \widetilde{v}_2$ for $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_2 | \alpha_1$.