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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 2 (October 21, 2015) —

Problem 1. Existence of Frobenius numbers.
If $A = \{a_1, \dots a_k\}$ is a finite set of positive numbers then $n \in \mathbb{N}$ is A-representable if and only if there exist $m_1, \dots, m_k \in \mathbb{N}_0$ such that $n = m_1a_1 + m_2a_2 + \dots + m_ka_k$. Moreover, $g(A) \in \mathbb{N}$ is called Frobenius number of A if and only if $g(A)$ is not A-representable while every integer $n > g(A)$ is A-representable.
Prove the following statements:

a) The Frobenius number of A does not exist if $\gcd(a_1, \dots, a_k) > 1$.
b) The Frobenius number of A does exist if $\gcd(a_1, \dots, a_k) = 1$.

Hint. Use the euclidean algorithm to decompose an integer n into $m_1a_1 + m_2a_2 + \dots + m_ka_k$ where $m_1, \dots, m_k \in \mathbb{Z}$ and $0 \le m_i < a_1$ for $i \in \{2, \dots, k\}$. Conclude $g(A) \le (a_1 - 1)(a_2 + a_3 + \dots + a_k)$.

Problem 2. A plethora of lattices.
The following lattices are important examples.

- The standard integer lattices $\mathbb{Z}^d \subset \mathbb{R}^d$ ($d \in \mathbb{N}$).
  $\mathbb{Z}^d := \left\{ (x_1, \dots, x_d)^\top \in \mathbb{R}^d \;\middle|\; x_i \in \mathbb{Z} \text{ for } i \in [d] \right\}$.
- The lattices $A_d \subset \mathbb{R}^{d+1}$ ($d \in \mathbb{N}$).
  $A_d := \mathbb{Z}^{d+1} \cap H$
  where H denotes the hyperplane $H := \left\{ (x_1, \dots, x_{d+1})^\top \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i\in[d+1]} x_i = 0 \right\}$.
- The lattice $B_d \subset \mathbb{R}^d$ ($d \in \mathbb{N}$).
  $B_d := \left\{ \sum_{i\in[d]} \lambda_i v_i \in \mathbb{R}^d \;\middle|\; \lambda_i \in \mathbb{Z} \text{ for all } i \in [d] \right\}$
  where $v_i := e_i - e_{i+1}$ for $i \in [d-1]$ and $v_d := e_d$.
- The lattices $D_d \subset \mathbb{R}^d$ ($d \in \mathbb{N}$).
  $D_d := \left\{ (x_1, \dots, x_d)^\top \in \mathbb{R}^d \;\middle|\; x_i \in \mathbb{Z} \text{ for } i \in [d] \text{ and } \sum_{i\in[d]x_i} \text{ is an even integer} \right\}$.
- The lattice E_8.
  $E_8 := D_8 \cup (D_8 + x_0)$
  where $x_0 = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)^\top \in \mathbb{R}^8$.
- The lattice E_7.
  $E_7 := E_8 \cap H$
  where H denotes the hyperplane $H := \left\{ (x_1, \dots, x_8)^\top \in \mathbb{R}^8 \;\middle|\; \sum_{i\in[8]} x_i = 0 \right\}$.
- The lattice E_6.
  $E_6 := E_8 \cap H$
  where H denotes the subspace $H := \left\{ (x_1, \dots, x_8)^\top \in \mathbb{R}^8 \;\middle|\; x_1 + x_8 = 0 \text{ and } \sum_{i=2}^{7} x_i = 0 \right\}$.

a) Check that one infinite family of A_d, B_d or D_d is indeed a lattice.
b) Draw pictures for D_2 and A_2.
c) Prove that $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is a basis of E_8 where $u_1 := 2e_1$, $u_i := e_i - e_{i-1}$ for $2 \le i \le 7$ and $u_8 := \frac{1}{2}\sum_{i\in[8]} e_i$.

Problem 3. Lattice points a ball.
Let $\Lambda \subset \mathbb{R}^d$ be a lattice with $\mathsf{rk}(\Lambda) = d$ and $B_\rho(x_0) := \left\{ x \in \mathbb{R}^d \;\middle|\; \|x = x_0\| \le \rho \right\}$ be the ball of radius ρ centred at x_0. Show that $B_\rho(x_0) \cap \Lambda$ is a finite set for all $\rho > 0$.

Problem 4. Change of a lattice basis.
Let $u_1, \dots, u_d$ be a basis of a lattice $\Lambda \subset \mathbb{R}^d$ and choose two distinct indices $i \ne j$.
Prove that $u_1, \dots, u_{i-1}, \widetilde{u}_i, u_{i+1}, \dots u_d$ is a lattice basis of Λ for each $\widetilde{u}_i := u_i + \alpha u_j$ with $\alpha \in \mathbb{Z}$.