TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

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Discrete Geometry: Lattice Polytopes (winter 2014/15)

— Problem Set 1 (October 14, 2015) —

Please bookmark:

`https://www-m10.ma.tum.de/bin/view/Lehre/WS1516/DGLPWS1516/WebHome`
or
`http://tinyurl.com/dg-lattice-polytopes`

This website provides important news for the lectures and tutorials. It will be updated regularly.

Problem 1. Pick's formula.
Let P be a lattice polygon in $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ with $I(P)$ interior lattice points and $B(P)$ lattice points on the boundary of P. Then Pick's formula states that $\mathsf{vol}(P) = I(P) + \frac{1}{2}B(P) - 1$.

Show that Pick's formula holds for every lattice triangle and complete the proof sketched in the lecture.

Problem 2. Non-unimodular tetrahedra?
Pick's formula implies that every lattice polygon in $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$ with precisely 3 lattice points has volume $\frac{1}{2}$. Phrased differently, all lattice polygons with precisely 3 lattice points are unimodular. Is it true that all lattice polytopes in $\Lambda = \mathbb{Z}^3 \subset \mathbb{R}^3$ with precisely four lattice points are unimodular?

Problem 3. Unimodular equivalent lattices.
Recall that lattices $\Lambda \subset \mathbb{R}^d$ and $\Lambda' \subset \mathbb{R}^{d'}$ are unimodular equivalent if and only if there exists a linear map $T : \mathsf{span}_{\mathbb{R}}\Lambda \to \mathsf{span}_{\mathbb{R}}\Lambda'$ that induces a bijection between Λ and Λ'.

a) Show that the lattices Λ and Λ' are unimodular equivalent if and only if there is a linear isomorphism $T : \mathsf{span}_{\mathbb{R}}\Lambda \to \mathsf{span}_{\mathbb{R}}\Lambda'$ that is represented by a matrix $M_T \in \mathbb{Z}^{d\times d}$ with $\det(M_T) = \pm 1$.
b) Let Λ and Λ' be unimodular equivalent lattices that are generated by bases $\mathcal{B}$ and $\mathcal{B}'$ and consider the parallelepiped Π and Π' spanned by $\mathcal{B}$ and $\mathcal{B}'$ in the euclidean vector spaces $\mathbb{R}^d$ and $\mathbb{R}^{d'}$. Prove or disprove $\mathsf{vol}_{\mathbb{R}^d}(\Pi) = \mathsf{vol}_{\mathbb{R}^{d'}}(\Pi')$.

Problem 4. A family of lattice polytopes.

a) Consider the lattice polytope $P_3 := \mathsf{cov}\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\} \subset \mathbb{R}^3$. Draw this polytope, provide a simple description of P_3 as a solution of a system of linear inequalities and determine the number of lattice points for P_3!
b) For each positive integer n consider the lattice polytope

$$P_n := \mathsf{cov}\left\{ \begin{pmatrix}\pi(1)\\ \vdots \\ \pi(n)\end{pmatrix} \;\middle|\; \pi \in S_n \right\} \subset \mathbb{R}^n$$

where S_n is the symmetric group on $\{1, 2, \ldots, n\}$.

Provide a simple description of P_n as a solution of a system of linear inequalities.