Exercise 1. (Meromorphic functions on a torus)

Use the Riemann-Hurwitz formula to show that a meromorphic function on a torus with degree 2 (the lowest possible degree) has four ramification points.

Exercise 2. (Definition of the Weierstrass $\wp$-function)

Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent and let $\Gamma = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$.

**Remark.** The Weierstrass $\wp$-function is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

The purpose of this exercise is to show that $\wp(z)$ is a well defined meromorphic function on $\mathbb{C}$.

(a) For any complex numbers $\omega$ and $z$ with $|\omega| > 2|z|$ show that

$$\left| \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right| < \frac{10|z|}{|\omega|^3}.$$

(b) Show that there is a number $c > 0$ such that

$$|x_1 \omega_1 + x_2 \omega_2| \geq c(|x_1| + |x_2|)$$

for all $x \in \mathbb{R}^2$.

(c) Show that the series

$$\sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

converges absolutely for $z \in \mathbb{C} \setminus \Gamma$. Furthermore, show that it converges to a meromorphic function on $\mathbb{C}$, and that the convergence is uniform on compact sets.

**Hint for (c):** Use (a) and (b) and: How many $(n_1, n_2) \in \mathbb{Z}^2$ are there with $|n_1| + |n_2| \leq m$?

Exercise 3. (Holiday project on covering spaces, the fundamental group, and the conformal classification of tori)

(a) Read chapter 9 (“Covering Spaces”) of Jänich’s *Topology* (or chapter 1 of Jost’s *Compact Riemann Surfaces*). We are only interested in a less general setting where all topological spaces are in fact Riemann surfaces and all maps are not only continuous but holomorphic.

(b) Let $M$ and $N$ be Riemann surfaces and suppose $\pi : M \to N$ is an (unbranched) holomorphic covering. Let $S$ be a simply connected Riemann surface and let $f : S \to N$ be a holomorphic map. Show that there is a holomorphic map $F : S \to M$ such that $f = \pi \circ F$.

(c) Show that two tori $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Gamma'$ (where $\Gamma$ and $\Gamma'$ are lattices) are conformally equivalent if and only if $\Gamma' = a \Gamma$ for some $a \in \mathbb{C}^*$.