Riemann Surfaces
Exercise Sheet 2

Exercise 1. (connected and path-connected)
(a) Show that a Hausdorff space $X$ is connected if and only if $X$ and $\emptyset$ are the only subsets that are simultaneously open and closed.
(b) Show that a path-connected topological space is connected.
(c) Let the function $f : \mathbb{R} \to \mathbb{C}$ be defined by $f(t) = r(t) e^{it}$, where $r(t) = \arctan(x) + \frac{\pi}{2}$, and let $S_1 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = \pi^2\}$. Show that $A = f(\mathbb{R}) \cup S_1 \cup \{0\}$ is connected but not path-connected.
(e) Show that a manifold is connected if and only if it is path-connected.

Exercise 2. (different incarnations of the Riemann sphere)
(a) Show that the chart maps for $S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

$\psi_1 : S^2 \setminus \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \to \mathbb{C}, \quad \psi_1(x) = \frac{1}{1 - x_3} (x_1 + i x_2),$

$\psi_2 : S^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \to \mathbb{C}, \quad \psi_2(x) = \frac{1}{1 + x_3} (x_1 - i x_2)$

are holomorphically compatible.
(b) Show that the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the sphere $S^2$ and the complex projective line $\mathbb{C}P^1$ are all conformally equivalent.
(c) Show that the holomorphic functions on $\hat{\mathbb{C}}$ are the constant functions and the meromorphic functions on $\hat{\mathbb{C}}$ are the rational functions. What are the biholomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$?

Exercise 3. (lattices and tori)
Let $\omega_1, \omega_2 \in \mathbb{C} \cong \mathbb{R}^2$ be linearly independent over $\mathbb{R}$, and let $\Gamma \subseteq \mathbb{C}$ be the additive subgroup of $\mathbb{C}$ generated by $\omega_1, \omega_2$:

$$\Gamma = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 = \{z \in \mathbb{C} \mid z = n_1 \omega_1 + n_2 \omega_2 \text{ for some } n_1, n_2 \in \mathbb{Z}\}.$$ 

Let $\mathbb{C}/\Gamma$ be the quotient space of $\mathbb{C}$ modulo the equivalence relation $\sim$ defined by $z_1 \sim z_2 \iff z_1 - z_2 \in \Gamma$.

(a) Show that 0 is not an accumulation point of $\Gamma$ and hence that $\Gamma$ is a discrete subset of $\mathbb{C}$.
(b) Let $\tilde{\Gamma} = \mathbb{Z} \tilde{\omega}_1 + \mathbb{Z} \tilde{\omega}_2$, where $\tilde{\omega}_1, \tilde{\omega}_2 \in \mathbb{C}$. Show that $\tilde{\Gamma} = \Gamma$ if and only if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that

$$\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$ 

(c) Show that $\mathbb{C}/\Gamma$ is a surface.
(d) Define a complex structure for $\mathbb{C}/\Gamma$ so that the canonical map $\mathbb{C} \to \mathbb{C}/\Gamma, z \mapsto z + \Gamma$ is holomorphic.