Riemann Surfaces

Exercise Sheet 1

(Function theory warmup)

**Exercise 1.** (taking roots)
For any $z_0 \in \mathbb{C} \setminus \{0\}$, $n$, show that there exist an open neighborhood $U$ of $z_0$ and a holomorphic function $w$ on $U$ such that $(w(z))^n = z$.

**Exercise 2.** (qualitative behavior near a zero of order $n$)
Let $f$ be a non-constant holomorphic function on a domain $U \subseteq \mathbb{C}$, and suppose $z_0 \in U$ is a zero of order $n \geq 1$ of $f$. Show:
(a) There is a holomorphic function $h$ defined on an open neighborhood $U_0 \subseteq U$ of $z_0$ that has a simple zero at $z_0$ and satisfies $f(z) = (h(z))^n$. (Tip: Use Exercise 1.)
(b) There is an open neighborhood $U_1 \subseteq U_0$ of $z_0$ such that $h$ maps $U_1$ biholomorphically onto an open disk $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ around zero.
(c) On $U_1$ the function $f$ takes every value in $D_r \setminus \{0\}$ precisely $n$ times.

**Exercise 3.** (Möbius transformations and cross-ratio)
Remember that given three points $z_1, z_2, z_3 \in \mathbb{C} = \mathbb{C} \cup \{\infty\}$ and three points $w_1, w_2, w_3 \in \mathbb{C}$, there is a unique Möbius transformation $f(z) = \frac{az + b}{cz + d}$ with $f(z_k) = w_k$.

The cross-ratio of four different complex numbers $z_1, z_2, z_3, z_4$ is defined by\(^1\)

$$\text{cr}(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$  

(Infinites are cancelled, e.g. $\text{cr}(-i, 0, i, \infty) = \frac{(-i-0)(i-\infty)}{(0-i)(\infty+i)} = \frac{(-i)(-\infty)}{(-i)(\infty)} = -1$.)

(a) Show that $\text{cr}(z_1, z_2, z_3, z_4)$ is the image of $z_1$ under the Möbius transformation that maps $z_2, z_3, z_4$ to 0, 1, $\infty$, respectively.
(b) Show that there exists a Möbius transformation that maps four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ to four points $w_1, w_2, w_3, w_4$ if and only if

$$\text{cr}(z_1, z_2, z_3, z_4) = \text{cr}(w_1, w_2, w_3, w_4).$$

**Exercise 4.** (How does the cross-ratio change if the arguments are permuted?)
(a) Show that the cross-ratio does not change if the arguments are permuted by a permutation in the group

$$\{((), (12)(34), (13)(24), (14)(23))\}.$$  

(b) For $q \in \mathbb{C} \setminus \{0, 1\}$, compute the six values

$$\text{cr}(q, 0, 1, \infty), \text{cr}(q, 1, \infty, 0), \text{cr}(q, \infty, 0, 1),$$

$$\text{cr}(q, 1, 0, \infty), \text{cr}(q, \infty, 1, 0), \text{cr}(q, 0, \infty, 1).$$  

(c) Now let $q = \text{cr}(z_0, z_1, z_2, z_3)$. For all 24 permutations $\sigma \in S_4$, find expressions for the cross-ratio $\text{cr}(z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ in terms of $q$.

\(^1\) The cross-ratio is often defined differently with respect to the order of the parameters. This is not essential, as Exercise 4 shows.