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Riemann Surfaces

Mini Lecture Notes

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Disclaimer. This is not (and not meant to be) a textbook. It is more like a brief protocol of what I think I did in the lectures. Proofs and diagrams are not included. Use at your own risk. I am sure this text is full of mistakes. Most of it was written in a hurry. If you find a mistake, you could do me favor by writing me an email.

Contents

Some literature on Riemann surfaces (not a complete list)

Textbooks

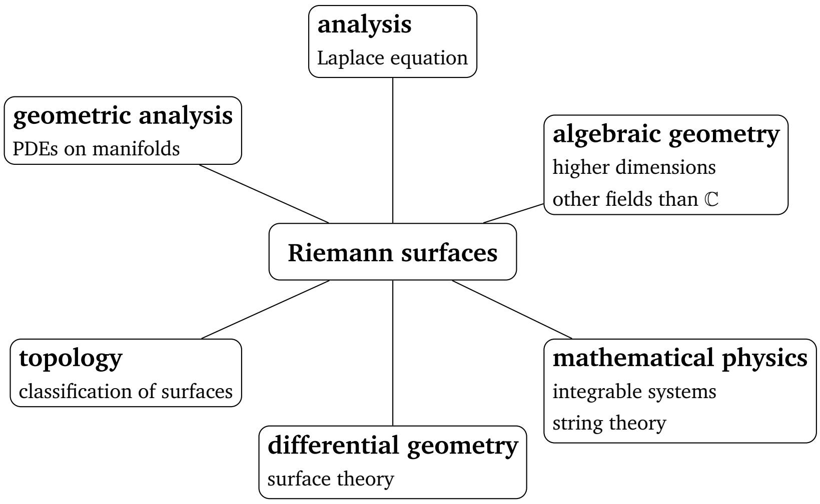
- Springer. *Introduction to Riemann surfaces*. Addison-Wesley, 1957.
- Lamotke. *Riemannsche Flächen*. Springer, 2005.
- Weyl. *Die Idee der Riemannschen Fläche*. Teubner, 1913.
- Miranda. *Algebraic Curves and Riemann Surfaces*. AMS, 1995.
- Jost. *Compact Riemann Surfaces*, 3rd ed. Springer, 2006.
- Farkas and Kra. *Riemann Surfaces*. Springer, 1980.
- Forster. *Riemannsche Flächen*. Springer, 1977.
- Griffiths and Harris. *Principles of Algebraic Geometry*. Wiley, 1994.

Other people's lecture notes

- Bobenko. *Compact Riemann Surfaces*. Lecture notes, available on `http://page.math.tu-berlin.de/~bobenko/skripte.html`.
- Donaldson. *Riemann Surfaces*. Lecture notes, 2004, available on `http://www2.imperial.ac.uk/~skdona`.
- Hitchin. *Geometry of Surfaces*, Sections 1-3. Lecture notes, 2004, available on `http://people.maths.ox.ac.uk/hitchin/hitchinnotes/hitchinnotes.html`.

Introduction

Overview



From the pendulum to Riemann surfaces

The concept of a Riemann surface originated from the study of *algebraic functions and their integrals*. Let us start with a simple physical system.

The pendulum equation is

$$\frac{d^2\alpha}{dt^2} = -\sin\alpha. \tag{1}$$

(The physical constants of the problem—the length of the pendulum and the acceleration due to gravity—have been absorbed into the unit of time.)

We want to solve this second order ODE to get the function $\alpha(t)$.

The energy

$$E = \frac{1}{2}\Big(\frac{d\alpha}{dt}\Big)^2 - \cos\alpha$$

is a constant of motion. Solve for $\frac{d\alpha}{dt}$ to reduce the problem to a first order ODE,

$$\frac{d\alpha}{dt} = \sqrt{2(E+\cos\alpha)}.$$

Separation of variables yields

$$t = \int \frac{d\alpha}{\sqrt{2(E+\cos\alpha)}}\,. \tag{2}$$

Now we have to

- solve the integral (2) to get $t(\alpha)$,
- invert the function $t(\alpha)$ to get $\alpha(t)$.

This is somewhat strange, because everyday experience with pendulums tells us that t is not a well defined function of α.

The substitution $u = \sin\frac{\alpha}{2}$ transforms the integral (2) to an integral of an algebraic function:

$$t = \int \frac{du}{\sqrt{(1-u^2)(\frac{E+1}{2} - u^2)}}\,.$$

For cosmetic reasons (see equation (5) below), we let $k^2 = \frac{E+1}{2}$,

$$t = \int \frac{du}{\sqrt{(1-u^2)(k^2-u^2)}}\,, \tag{3}$$

and perform one further substitution, $u = kx$, to obtain

$$t = \int \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}}. \tag{4}$$

It turns out that this integral cannot be solved in terms of elementary functions. Instead, one needs so-called *elliptic functions*. In fact, the Jacobi-sn-function (pronounced '*es-en*') happens to be defined by

$$x = \operatorname{sn}(t,k) \quad \text{means} \quad t = \int_0^x \frac{d\xi}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}. \tag{5}$$

So this is the solution of the pendulum equation (1). Of course, giving an unknown function a name does not solve any problem. If we introduce new named functions, we need to understand why we have to, introduce as few as reasonable, understand their interrelationships, and be able to compute them. The theory of Riemann surfaces can do that.

To see what this has to do with surfaces, let us look at some algebraic functions that we can integrate, and some that we cannot.

(1) $\int p(x)\,dx$, where $p(x) = a_0 + a_1x + \ldots + a_nx^n$ is a polynomial is easy to solve.

(2) $\int R(x)\,dx$, where $R(x) = \frac{p(x)}{q(x)}$ is a rational function, can be solved by partial fraction decomposition.

(3) $\int R\big(x, \sqrt{x-a}\big)\,dx$, where $R(x,y)$ is a rational function of x and y, can be reduced to (2) by the substitution $y = \sqrt{x-a}$.

(4) $\int R\big(x, \sqrt{(x-a)(x-b)}\big)\,dx$, where $a \neq b$, can be reduced to case (2) by the substitution $y = \sqrt{\frac{x-a}{x-b}}$.

(5) But $\int R\big(x, \sqrt{(x-a)(x-b)(x-c)}\big)\,dx$, where a, b, c are different, cannot be solved in elementary functions.

(6) More generally, $\int R\big(x, \sqrt{p(x)}\big)\,dx$, where p is a polynomial with different roots and $\deg(p) > 2$, cannot be solved in elementary functions.

So if $p(x)$ is a polynomial with different roots and $\deg(p) \leq 2$, then $\int R(x, \sqrt{p(x)})\,dx$ can be solved in elementary functions, but if $\deg(p) > 2$, it cannot.

Is there a deeper reason for this?

Consider the functions $\sqrt{p(x)}$ for complex x.

- The expression $\sqrt{x-a}$ has two values for every $x \in \mathbb{C}$ except a. We cannot choose one of the values consistently everywhere on $\mathbb{C}$ to obtain a holomorphic function. But if we cut the Riemann sphere along a path from a to ∞, we *can* choose one of the values consistently and obtain a holomorphic function on the cut sphere. Take another copy of the cut sphere and choose the opposite value there. Now glue the two spheres along the cuts where the functions take the same value. The result is the *Riemann surface* of the function $\sqrt{x-a}$. It is topologically again a sphere.
- For $\sqrt{(x-a)(x-b)}$ we proceed similarly, except that we cut from a to b. Again, the Riemann surface we obtain is topologically a sphere.
- To define $\sqrt{(x-a)(x-b)(x-c)}$ as a holomorphic function, we need to cut for example from a to b and from c to ∞. If we take two such spheres that take opposite values of the square root and glue them along cuts where the values coincide, we get a surface that is topologically a torus.
- The case of $\sqrt{(x-a)(x-b)(x-c)(x-d)}$ is similar, except we cut, e.g., from a to b and from c to d. Again we obtain a topological torus.
- In general, the Riemann surface for $\sqrt{p(x)}$, where p is a polynomial of degree $2n$ or $2n-1$ with different roots, is obtained by gluing two spheres with n cuts. This is topologically a surface of genus $n-1$, that is, a sphere with $n-1$ handles.

The fundamental difference between $\deg(p) \leq 2$ and $\deg(p) > 2$ turns out to be that the Riemann surface for $\sqrt{p(x)}$ is a sphere in the first case and a higher genus surface in the second.

We started with the motion of a pendulum and ended with the topological classification of surfaces.

Riemann surfaces: definition and examples

Definitions: topology, manifolds, Riemann surfaces

The upshot: A surface is locally like $\mathbb{R}^2$, and a Riemann surface is locally like $\mathbb{C}$.

A *topological space* is a set X together with a set $\mathscr{T}$ of subsets of X, so that:

- $\emptyset \in \mathscr{T}$ and $X \in \mathscr{T}$.
- If $U_1, U_2 \in \mathscr{T}$ then $U_1 \cap U_2 \in \mathscr{T}$.
- If $U_i \in \mathscr{T}$ for all $i \in I$, then $\bigcup_{i\in I} U_i \in \mathscr{T}$.

The subsets $U \in \mathscr{T}$ are *open*, their complements $X \setminus U$ are *closed*. $\mathscr{T}$ is called the *topology* of X.

A map $f : X \to Y$ between topological spaces is *continuous* if the preimages of open sets in Y are open in X. If f is also bijective with continuous inverse, then f is a *homeomorphism*, and the spaces X and Y are *homeomorphic*.

A topological space is *compact* if every open cover has a finite sub-cover.

A topological space X is *connected* if the following holds: If U_1 and U_2 are disjoint open sets in X and $U_1 \cup U_2 = X$, then $U_1 = X$ and $U_2 = \emptyset$, or $U_1 = \emptyset$ and $U_2 = X$.

X is *path-connected* if for all $x, y \in X$ there is a continuous function $f : [0,1] \to X$ with $f(0) = x$, $f(1) = y$.

Any subset $A \subseteq X$ is itself a topological space with the induced *subspace topology*: The open sets of A are the intersections of open sets of X with A.

A topological space X is a *Hausdorff space* if for any two distinct points $x, y \in X$ there are disjoint open sets U and V containing x and y, respectively.

A topological space X *satisfies the second axiom of countability*, or *is second countable*, if there is a countable subset $\mathscr{B} \subseteq \mathscr{T}$ such that for all $U \in \mathscr{T}$, $U = \bigcup_{V\in\mathscr{B}, V\subseteq U} V$. For example, $\mathbb{R}^n$ is second countable: Let $\mathscr{B}$ be the set of open balls with rational radius and rational center coordinates.

Let $\sim$ be an equivalence relation on X, let $X/\!\sim$ be the set of equivalence classes, and let $p : X \mapsto X/\!\sim$ be the canonical map. The *quotient topology* of $X/\!\sim$ is defined by

$$U \subseteq X/\!\sim \quad \text{is open} \quad \Leftrightarrow \quad p^{-1}(U) \subseteq X \quad \text{is open.}$$

Example. Square with pairs of opposite sides glued together.

An *n-dimensional [topological] manifold* is a Hausdorff space with the property that every point has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. To exclude pathological examples, it is also required that M is second countable.

A manifold is connected if and only if it is path-connected (exercise).

A *[topological] surface* is a 2-dimensional manifold.

Remark. In the case of surfaces, requiring the second axiom of countability is equivalent to requiring that the surface can be triangulated.[1]

Now let M be a surface.

A *complex chart* (U, z) of M is an open set $U \subseteq M$ together with a map $z : U \to \mathbb{C}$, which maps U homeomorphically onto an open set $z(U) \subseteq \mathbb{C}$. The set U is called a chart neighborhood.

Two complex charts (U_1, z_1), (U_2, z_2) are called *homeomorphically compatible* if $U_1 \cap U_2 = \emptyset$ or otherwise if the function

$$z_2 \circ z_1^{-1} : \mathbb{C} \supseteq z_1(U_1 \cap U_2) \to z_2(U_1 \cap U_2) \subseteq \mathbb{C}$$

is biholomorphic. The map $z_2 \circ z_1^{-1}$ is called the *chart transition map*.

A *holomorphic atlas* for M is a set $\mathscr{A} = \{(U_i, z_i)\}_{i\in I}$ of compatible holomorphic charts such that the chart neighborhoods U_i cover M.

[1] Radó, Über den Begriff der Riemannschen Fläche, *Acta Sci. Math. (Szeged)* 2:2 (1925).

Two holomorphic atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are *compatible*, if their union $\mathscr{A}_1 \cup \mathscr{A}_2$ is also a holomorphic atlas, i.e., if all charts of $\mathscr{A}_1$ are holomorphically compatible with all charts of $\mathscr{A}_2$. This defines an equivalence relation for holomorphic atlases.

A *complex structure* for M is an equivalence class of compatible holomorphic atlases. A complex structure for M is therefore determined by one holomorphic atlas for M. Every complex structure contains a unique maximal atlas: the union of all atlases in the equivalence class.

A *Riemann surface* is a connected surface together with a complex structure.

A map $f : M \to N$ between Riemann surfaces is called *holomorphic* if, for all charts (U,z) of M and (V,w) of N, the map $w \circ f \circ z^{-1}$ is holomorphic (as a map between open subsets of $\mathbb{C}$) wherever it is defined.

If a holomorphic map $f : M \to N$ is bijective, then the inverse f^{-1} is also holomorphic, and f is called *biholomorphic* or *conformal*. The Riemann surfaces M and N are then called *conformally equivalent*.

The simplest examples of Riemann surfaces

- The *complex plane* $\mathbb{C}$, with an atlas consisting only of one chart, $id : \mathbb{C} \to \mathbb{C}$.

Holomorphic maps $M \to \mathbb{C}$ are also called *holomorphic functions* on M.

- Every open subset of $\mathbb{C}$, in particular
  - the *unit disk* $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$,
  - the *upper half plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$,
  - the *punctured complex plane* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

In general, every open subset of a Riemann surface is a Riemann surface.

- The *extended complex plane* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The topology is defined by

$$U \subseteq \widehat{\mathbb{C}} \quad \text{open} \quad :\Longleftrightarrow \quad \begin{cases} \infty \notin U \text{ and } U \text{ open in } \mathbb{C} \\ \text{or} \\ \infty \in U \text{ and } \widehat{\mathbb{C}} \setminus U \subseteq \mathbb{C} \text{ is compact.} \end{cases}$$

  The complex structure is defined by an atlas consisting of two charts:

$$\phi_1 : \widehat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C} \longrightarrow \mathbb{C}, \quad \phi_1(p) = p \qquad \text{and} \qquad \phi_2 : \widehat{\mathbb{C}} \setminus \{0\} \longrightarrow \mathbb{C}, \quad \phi_2(p) = \begin{cases} \frac{1}{p} & \text{if } p \neq \infty \\ 0 & \text{if } p = \infty \end{cases}$$

  They are holomorphically compatible: $\phi_2 \circ \phi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*, \quad p \mapsto \frac{1}{p}$.

Holomorphic maps $M \to \widehat{\mathbb{C}}$ are also called *meromorphic functions* on M.

- The *sphere* $S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\} \subseteq \mathbb{R}^3$ with the subspace topology induced by $\mathbb{R}^3$ and the complex structure defined by the atlas consisting of two charts

$$\psi_1 : S^2 \setminus \left\{ \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \right\} \to \mathbb{C}, \quad \psi_1(x) = \frac{x_1 + i x_2}{1 - x_3} \quad \text{and} \quad \psi_2 : S^2 \setminus \left\{ \left(\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right) \right\} \to \mathbb{C}, \quad \psi_2(x) = \frac{x_1 - i x_2}{1 + x_3}.$$

- The *complex projective line* (aka *1-dimensional complex projective space*) $\mathbb{CP}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$, where

$$u \sim v \quad :\Leftrightarrow \quad u = \lambda v \quad \text{for some} \quad \lambda \in \mathbb{C}^*.$$

  The equivalence classes, i.e., the points of $\mathbb{CP}^1$, correspond to 1-dimensional subspaces of the complex vector space $\mathbb{C}^2$. The equivalence class of $u = (u_1, u_2)$ is $[u] := \mathbb{C}^* u$, and one usually writes $[u_1, u_2]$ instead of $[(u_1, u_2)]$. The topology of $\mathbb{CP}^2$ is the quotient topology. The complex structure is determined by the atlas consisting of the two charts

$$\chi_1 : U_1 \to \mathbb{C}, \quad \chi_1(u) = \frac{u_2}{u_1} \quad \text{where} \quad U_1 = \{[u_1, u_2] \in \mathbb{CP}^1 \mid u_1 \neq 0\},$$
$$\chi_2 : U_2 \to \mathbb{C}, \quad \chi_2(u) = \frac{u_1}{u_2} \quad \text{where} \quad U_2 = \{[u_1, u_2] \in \mathbb{CP}^1 \mid u_2 \neq 0\}.$$

The Riemann surfaces $\widehat{\mathbb{C}}$, S^2 and $\mathbb{CP}^1$ are all conformally equivalent. They are just different incarnations of the Riemann sphere.

In fact, all Riemann surfaces that are homeomorphic to the sphere are conformally equivalent. (To be proved later. Follows from the Riemann-Roch theorem.)

Algebraic curves

Let $p \in \mathbb{C}[z,w]$ be a non-constant polynomial in two indeterminates. The set of zeros of p,

$$C = \{(z,w) \in \mathbb{C}^2 \,|\, p(z,w) = 0\},$$

is called an *[affine] algebraic curve*. If $dp_{(z,w)} = \big(\partial_1 p(z,w), \partial_2 p(z,w)\big) \neq 0$ for all $(z,w) \in C$ then the algebraic curve C is called *regular* (or *smooth*), otherwise it is called *singular*.

A regular algebraic curve is equipped with a natural complex structure, which can be described as follows. The implicit function theorem says:

(1) If $(z_1, w_1) \in C$ and $\partial_2 p(z_1, w_1) \neq 0$, then there exist open neighborhoods $U_1 \ni z_1$ and $V_1 \ni w_1$ and a holomorphic function $f : U_1 \to V_1$, such that

$$C \cap (U_1 \times V_1) = \text{“graph of } f\text{”} = \big\{\big(z, f(z)\big)\big\}_{z \in U_1}.$$

The projection $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$, $\pi_1(z,w) = z$ maps $C \cap (U_1 \times V_1)$ homeomorphically onto U_1.

(2) If $(z_2, w_2) \in C$ and $\partial_1 p(z_2, w_2) \neq 0$, then there exist open neighborhoods $U_2 \ni z_2$ and $V_2 \ni w_2$ and a holomorphic function $g : V_2 \to U_2$, such that

$$C \cap (U_2 \times V_2) = \text{“graph of } g\text{”} = \big\{\big(g(w), w\big)\big\}_{w \in V_2}.$$

The projection $\pi_2 : \mathbb{C}^2 \to \mathbb{C}$, $\pi_2(z,w) = w$ maps $C \cap (U_2 \times V_2)$ homeomorphically onto V_2.

Thus, C is a surface: it is a second countable Hausdorff space (as subspace of $\mathbb{C}^2$) and locally homeomorphic to $\mathbb{R}^2 \cong \mathbb{C}$, via the maps of the sort

$$\pi_1|_{C \cap (U_1 \times V_1)} : C \cap (U_1 \times V_1) \to \mathbb{C} \quad \text{and} \quad \pi_2|_{C \cap (U_2 \times V_2)} : C \cap (U_2 \times V_2) \to \mathbb{C}$$

described above. Further, these maps form a holomorphic atlas. If they involve the same projection, like, for example, $\pi_1|_{C \cap (U_1 \times V_1)}$ and $\pi_1|_{C \cap (\tilde{U}_1 \times \tilde{V}_1)}$, they are holomorphically compatible because the chart transition map

$$\pi_1|_{C \cap (\tilde{U}_1 \times \tilde{V}_1)} \circ (\pi_1|_{C \cap (U_1 \times V_1)})^{-1}$$

is the identity on $U_1 \cap \tilde{U}_1$. If they involve different projections, they are holomorphically compatible because the chart transition map

$$\pi_2 \circ (\pi_1|_{C \cap (U_1 \times V_1)})^{-1}(z) = \pi_2(z, f(z)) = f(z)$$

is holomorphic.

So C is a surface equipped with a complex structure. If C is connected, it is a Riemann surface.

Theorem. *A regular algebraic curve $p(z,w) = 0$ is connected if and only if p is irreducible (i.e. not the product of two non-constant polynomials).*

Proof. “⇒”: Exercise. “⇐”: Involves more algebraic machinery than I am willing to go into. □

Elliptic and hyperelliptic curves

Let $p(z) = a(z - z_1)(z - z_2)\ldots(z - z_n)$ be a complex polynomial of one indeterminate, $n = \deg(p)$, let

$$f(z,w) = w^2 - p(z),$$

and consider the algebraic curve

$$C = \{(z,w) \in \mathbb{C}^2 \,|\, f(z,w) = 0\}.$$

Since $\partial_2 f(z,w) = 2w$ and $\partial_1 f(z,w) = p'(z)$, the curve C is regular if and only if the zeros $z_1, \ldots, z_n$ are all distinct. Assume this is the case.

$n = 1$: The curve C is conformally equivalent to $\mathbb{C}$ because w is a global coordinate.
$n = 2$: The curve C is conformally equivalent to $\mathbb{C}^*$. (Exercise.)
$n \in \{3,4\}$: The curve C is called an *elliptic curve*.
$n \geq 5$: The curve C is called a *hyperelliptic curve*.

Let us try to understand the topology of the curve C. For each value of $z \in \mathbb{C}$, the curve C contains two points $(z, w) = (z, \sqrt{p(z)})$, except if z is a zero of p; then there is only one point $(z, 0)$.

Consider first the case that $n = 2m$ is even. Connect the zeros $z_1, \dots, z_{2m}$ in pairs by disjoint simple curves. On the complement of these curves, we can define a holomorphic function $w_1(z) = \sqrt{p(z)}$. It changes sign if we cross a curve. Take another copy of the complex plane with the same curves, on which the function $w_2 = -w_1(z)$ is defined. Now cut the complex planes open along the curves and glue them together such that w_1 and w_2 fit together to yield a holomorphic function defined on the glued surface. Consider each copy of $\mathbb{C}$ as Riemann sphere $\hat{\mathbb{C}}$ minus one point ∞. Then the result of the gluing operation is a compact surface of genus $g = m - 1$ (a sphere with g handles) minus two points ∞_1, ∞_2.

Not consider the case where $n = 2m - 1$ is odd. The construction proceeds as in the even case, except that we connect $2m - 2$ of the roots in pairs and the remaining one with ∞. After gluing we obtain a compact surface of genus g minus one point ∞ (because the two ∞s are glued together).

It makes sense to compactify the curve C by adding the "infinite point(s)".

Compactification of elliptic and hyperelliptic curves

• Consider first the case that $n = 2m$ is even. Consider the elliptic or hyperelliptic curve

$$\tilde{C} = \{(u, v) \in \mathbb{C}^2 \mid v^2 - q(u) = 0\},$$

where q is the polynomial

$$q(u) = u^{2m}\, p\left(\tfrac{1}{u}\right) = (1 - z_1 u)\dots(1 - z_{2m} u).$$

The curve $\tilde{C}$ has two points (u, v) with $u = 0$, the points $(u, v) = (0, \pm 1)$. Let $\tilde{C}^* = \tilde{C} \setminus \{(0, \pm 1)\}$.

The curve C contains one or two points (z, w) with $z = 0$, depending on whether or not $p(0) = 0$. Let $C^* = C \setminus \{(z, w) \mid z = 0\}$.

The punctured curves C^* and $\tilde{C}^*$ are conformally equivalent: the map $C^* \longrightarrow \tilde{C}^*$,

$$(z, w) \longmapsto (u, v) = \left(\frac{1}{z}, \frac{w}{z^m}\right)$$

is biholomorphic with inverse $(u, v) \mapsto (z, w) = \left(\frac{1}{u}, \frac{v}{u^m}\right)$.

Now glue the surfaces C and $\tilde{C}$ together by identifying corresponding points. The result is a compact Riemann surface

$$\widehat{C} = (C \dot{\cup} \tilde{C})/\sim$$

with injective holomorphic maps $C \hookrightarrow \widehat{C}$ and $\tilde{C} \hookrightarrow \widehat{C}$ whose images cover $\widehat{C}$. We identify C and $\tilde{C}$ with their images in $\widehat{C}$. Then $\widehat{C} \setminus C$ contains two points: the points $(u, v) = (0, \pm 1)$ of $\tilde{C}$.

• The case that $n = 2m - 1$ is odd works similarly, except that

$$q(u) = u^{2m} p\left(\tfrac{1}{u}\right) = u(1 - z_1 u)\dots(1 - z_n u).$$

Now $q(u) = 0$, so $\tilde{C}$ contains only one point (u, v) with $u = 0$, namely $(u, v) = (0, 0)$. Define $\tilde{C}^* = \tilde{C} \setminus \{(0, 0)\}$. The curve C^*, the biholomorphic map $C^* \to \tilde{C}^*$, and the compact Riemann surface $\widehat{C}$ are then defined as in the previous case. Now $\widehat{C} \setminus C$ contains only one point, the point $(u, v) = (0, 0)$ of $\tilde{C}$.

Thus, a hyperelliptic curve can be compactified by adding one (if $n = 2m - 1$ odd) or two (if $n = 2m$ even) points at infinity. The result is a compact surface of genus $m - 1$, i.e., a sphere with $m - 1$ handles.

Remarks. There is a way to turn any algebraic curve, regular or not, into a compact Riemann surface. This is called *normalization* of algebraic curve. When one considers algebraic curves as Riemann surfaces, one is usually interested in the compact Riemann surfaces obtained after compactification/normalization.

Theorem. *Every compact Riemann surface is realized as a (normalized/compactified) algebraic curve.*

(Without proof (for now).)

Orbit spaces of holomorphic group actions

Let $U \subseteq \mathbb{C}$ be a domain, let G be a group of holomorphic maps $U \mapsto U$, and let $\sim$ be the equivalence relation

$$u_1 \sim u_2 \quad :\Longleftrightarrow \quad \exists g \in G : u_2 = g(u_1).$$

The equivalence class of a point $u \in U$ is the *orbit* $Gu = \{g(u)\}_{g\in G}$ of u under the action of G. The quotient space $U/\sim$ is usually denoted by U/G. We want to turn U/G into a Riemann surface such that the canonical map

$$p : U \to U/G, \quad p(u) = Gu$$

is holomorphic. For this to work, the following two conditions have to be satisfied:

(1) Every point $u \in U$ has an open neighborhood $V \subseteq U$ such

$$g(V) \cap V = \emptyset \quad \text{for all} \quad g \in G \setminus \{id_U\}. \tag{$*$}$$

This implies in particular that the maps $g \in G \setminus \{id_U\}$ do not have fixed points: $g(u) = u \Rightarrow g = id_U$. If this is the case, one says G *acts freely* on U.

(2) Any two different points $u_1, u_2 \in U$ have open neighborhoods $V_1 \ni u_1$, $V_2 \ni u_2$, such that

$$V_1 \cap g(V_2) = \emptyset.$$

Condition (2) implies that the quotient space U/G is a Hausdorff space. Condition (1) allows us to define complex charts on U/G as follows: Suppose an open subset $V \subseteq U$ satisfies ($*$). Consider $\mathscr{V} = p(V)$, the set of orbits $Gu \in U/G$ of points $u \in V$. The set $\mathscr{V}$ is open in the quotient topology of U/G because $p^{-1}(\mathscr{V}) = \bigcup_{g\in G} g(V)$ is a union of open sets. Any orbit $X \in \mathscr{V}$ intersects V in a single point. This follows from ($*$). We can therefore define the chart maps $z_{\mathscr{V}} = \mathscr{V} \to \mathbb{C}$ by $X \cap V = \{z_{\mathscr{V}}(X)\}$. The chart transition maps $z_{\mathscr{V}_2} \circ (z_{\mathscr{V}_1})^{-1}$ (where $\mathscr{V}_k = p(V_k)$ and $V_k \subset U$ are open subsets satisfying ($*$)) are holomorphic (exercise). Thus the charts $(\mathscr{V}, z_{\mathscr{V}})$ form a holomorphic atlas and equip U/G with a conformal structure. This is the unique conformal structure in which the canonical map p is holomorphic (exercise).

Examples. (a) $\mathbb{C}/G$, where $G = \omega\mathbb{Z}$ for $\omega \in \mathbb{C}^*$, or, more precisely, $G = \{z \mapsto z + b\}_{b\in\omega\mathbb{Z}}$.
(b) The tori $G = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, where $\omega_k \in \mathbb{C}^*$, $\omega_2/\omega_1 \notin \mathbb{R}$.

Theorem. *Every Riemann surface is conformally equivalent to either $\widehat{\mathbb{C}}$, $\mathbb{C}/G$, or $\mathbb{D}/G$, where G is a group of Möbius transformations that acts freely on $\mathbb{C}$ or $\mathbb{D}$, respectively.*

(Without proof.)

Surfaces with Riemannian metric

Let M be a 2-dimensional Riemannian manifold, i.e. a smooth surface with Riemannian metric g. A chart (U, x), $x : U \to \mathbb{R}^2$ is called *conformal* if $g = e^{2u}(dx_1^2 + dx_2^2)$ where $u \in C^\infty(M)$.

Theorem (Korn-Liechtenstein)**.** *Every point of a 2-dimensional Riemannian manifold has a neighborhood on which conformal coordinates exist.*

(Without proof.)

An atlas consisting of conformal charts defines a complex structure on M.

Polyhedral surfaces

A *polyhedral surface* M is as surface obtained by gluing euclidean polygons along edges. Every point except the vertices has a neighborhood that can be mapped isometrically to $\mathbb{R}^2 \cong \mathbb{C}$. Now consider a vertex v. Take a metric disk around v, cut it open along a radius, develop the cut disk isometrically to the plane with the vertex at 0 and apply the map $z \mapsto z^\alpha$ where $\alpha = 2\pi/\Theta$ and Θ is the angle sum around v. These maps $M \to \mathbb{C}$ are holomorphically compatible and define a complex structure on the polyhedral surface.

Holomorphic maps between Riemann surfaces: General properties

Let $f : M \to N$ be a non-constant holomorphic map between Riemann surfaces N and M (as defined in Lecture 2).

Lemma (and Definition). *For each point $p \in M$ there are charts (U, ϕ) of M and (W, ψ) of N such that $p \in U$, $\phi(p) = 0$, and $\psi \circ f \circ \phi^{-1}(z) = z^k$ for some integer $k \geq 1$. The integer k is unique. We denote it by $k_p(f)$ and call it the* ramification index *of f at p.*

The number $k_p(f)$ can be understood as the *multiplicity* with which f takes its value at p. It can be defined in purely topological terms as follows: The point p has a neighborhood U such that the function f attains the value $f(p)$ exactly once in U, and it attains any other value $f(u)$ for $u \in U \setminus \{p\}$ exactly $k_p(f)$ times in U.

(Due to a temporary suspension of of subway services to Garching following a power failure in Munich, this lecture was shorter than usual.)

If $k_p(f) > 1$, then p is called a *ramification point* of f, and $f(p)$ is called a *branch point*. (In German there is only one word: *Verzweigungspunkt*.)

Corollary 1. *The function f is an open mapping (i.e., the images of open maps are open).*

Corollary 2 (maximum principle)**.** *Let $h : M \to \mathbb{C}$ be a holomorphic function. If $|h|$ or $\operatorname{Re} h$ or $\operatorname{Im} h$ attains a local maximum or minimum in M, then h is constant.*

Corollary 3. *If M is compact, then f is surjective and hence N is also compact.*

A *surjective* map between surfaces that looks in the neighborhood of every point and in suitable coordinates like $z \mapsto z^k$ ($k \in \mathbb{N}$) is called a *branched covering*.

Corollary 4 (Fundamental theorem of algebra)**.** *Every non-constant polynomial has a zero.*

Corollary 5. *A holomorphic function $M \to \mathbb{C}$ on a compact Riemann surface M is constant.*

Corollary 6. *The set of ramification points is discrete. If M is compact, the set of ramification points is finite.*

Corollary 7. *For all $y \in N$, the preimage $f^{-1}(\{y\}) \subseteq M$ is discrete. If M is compact, it is finite.*

From now on, let M be compact. For $q \in N$, let

$$d(y) = \sum_{x \in f^{-1}(\{y\})} k_x(f).$$

That is, $d(y)$ is the number of times that f takes the value y, counting multiplicities.

Corollary 8. *The function $d : N \to \mathbb{Z}$ is constant.*

The constant value of d is called the *degree of f* and denoted $\deg(f)$. The function f is called a $\deg(f)$*-sheeted branched covering* of N.

Examples

- The degree of a polynomial $p \in \mathbb{C}[z]$, considered as a holomorphic map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, is just the polynomial degree of p. The ramification points are the zeros of p', with ramification index equal to $m+1$ if m is the order of the zero of p', and the point ∞ with ramification order $\deg(p)$.
- Let M be the (compactified) elliptic or hyperelliptic curve

  $$w^2 = p(z), \quad \text{where} \quad p(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

  The holomorphic map $z : M \to \widehat{\mathbb{C}}$ has degree 2; it is a 2-sheeted branched covering of $\widehat{\mathbb{C}}$. The points are $(z, w) = (z_k, 0) \in M$ for $k = 1, \ldots, n$ are ramification points with ramification index 2. If n is odd $\infty_1 \in M$, the preimage of $\infty \in \widehat{\mathbb{C}}$, is also a ramification point with index 2. (Any 2-sheeted branched covering can only have ramification points of index 2.) The branch points are the $z_k \in \widehat{\mathbb{C}}$ and, if n odd, also ∞. If n is even, then $\infty \in \widehat{\mathbb{C}}$ has two preimages $\infty_1, \infty_2 \in M$ and f is unramified at $\infty_{1,2}$.

  What are the ramification points of w and what are their ramification indices? (Exercise.)

The classification of compact surfaces, the Euler characteristic, and the Riemann-Hurwitz formula[1]

In this section, we distinguish surfaces only up to homeomorphism. That is, homeomorphic surfaces are considered the same.

The *sphere* S^2 can also be obtained by gluing a digon according to the scheme aa^{-1}.

The *torus* T^2 is the surface obtained by rotating a circle around a line in the same plane that does not intersect the circle. The same surface is obtained by gluing the sides of a rectangle according to the scheme $aba^{-1}b^{-1}$.

The *[real] projective plane* $\mathbb{R}\mathrm{P}^2$ is (topologically) the sphere S^2 with opposite points identified: $\mathbb{R}\mathrm{P}^2 = S^2/\{\pm 1\}$. $\mathbb{R}\mathrm{P}^2$ is also the closed disk with opposite points on the boundary identified, and a Möbius band with a disk glued to its boundary.

A surface is *orientable* if there exists an atlas (of topological charts) such that all chart transition maps are orientation preserving. Otherwise the surface is *nonorientable*. Every Riemann surface is orientable, because the chart transition maps of a holomorphic atlas are orientation preserving. The projective plane and the Möbius band are examples of nonorientable surfaces. Another example is the *Klein bottle*, which is a rectangle with sides glued according to $aba^{-1}b$. It can be shown that a surface is nonorientable if and only if it contains a Möbius band.

The *connected sum* $M\#N$ of two surfaces M, N is the surface obtained by removing disks from M and N, then gluing the surfaces together along their boundary. Since we distinguish surfaces only up to homeomorphism, it does not matter which disks are removed and (a more subtle point) how the boundaries are glued. The connected sum operation is commutative and associative. Taking the connected sum with a sphere does not change the surface: $S^2\#M \cong M$.

For $g = 0, 1, 2, \ldots$, the surface

$$S^2 \underbrace{\#T^2\#T^2\#\ldots\#T^2}_{g \text{ times}} \overset{g\geq 1}{=} \underbrace{T^2\#T^2\#\ldots\#T^2}_{g \text{ times}}$$

is called a *sphere with g handles*. A *handle* is a torus with a disk removed.

For $g = 1, 2, 3, \ldots$, the surface

$$\underbrace{\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2\#\ldots\#\mathbb{R}\mathrm{P}^2}_{g \text{ times}}$$

is called a *sphere with g crosscaps*. A *crosscap* is a projective plane with a disk removed. (The name is due to the shape of a particular realization in $\mathbb{R}^3$ as a surface with self-intersections.) A *crosscap* is the same as a *Möbius band*.

Theorem (Classification of compact surfaces)**.** *Every orientable compact surface is homeomorphic to a sphere with g handles for a unique $g \in \{0, 1, 2, \ldots\}$. Every nonorientable compact surface is homeomorphic to a sphere with g crosscaps for a unique $g \in \{1, 2, 3, \ldots\}$.*

The number g is called the *genus* of the surface. Thus, the classification theorem says that the topological type of a surface is completely determined by its orientability status and its genus.

(Incidentally, the genus is the maximal number of disjoint simple closed curves along which one can cut a surface without separating it into different components.)

[1]A more detailed exposition can be found for example in chapter 1 of J. Stillwell, *Classical Topology and Combinatorial Group Theory* (2nd ed.), GTM 72, Springer, 1993. A different proof of the classification theorem, which does not use the normal form, is explained in: G. K. Francis and J. R. Weeks, Conway's ZIP proof, *Amer. Math. Monthly* 106:5 (1999), 393–399. Making these proofs rigorous, i.e., eliminating any need to appeal to intuition, requires a lot more work; see for example chapter 9 in: M. Hirsch, *Differential Topology*, GTM 33, Springer, 1976.

Theorem (Classification of compact surfaces, normal form version)**.** *Every orientable compact surface is either homeomorphic to a digon with sides glued according to the scheme aa^{-1} [the sphere], or to a $4g$-gon with sides glued according to the scheme*

$$a_1 b_1 a_1^{-1} b_1^{-1}\, a_2 b_2 a_2^{-1} b_2^{-1} \,\dots\, a_g b_g a_g^{-1} b_g^{-1}$$

[a sphere with g handles] for a unique $g \geq 1$. Every nonorientable compact surface is homeomorphic to a $2g$-gon with sides glued according to the scheme

$$a_1 a_1\, a_2 a_2 \,\dots\, a_g a_g$$

[a sphere with g crosscaps] for a unique $g \geq 1$.

Theorem. *Let v, e, and f be the numbers of vertices, edges, and faces of a polygonal decomposition of a compact surface M. Then the number $\chi_M = v - e + f$ depends only on the topological type of the surface. For an orientable surface of genus g, $\chi_M = 2-2g$. For a non-orientable surface homeomorphic to a sphere with g crosscaps, $\chi_M = 1 - g$.*

Theorem (Riemann-Hurwitz)**.** *Let $f : M \to N$ be a nonconstant holomorphic map between compact Riemann surfaces. Then the Euler characteristic of M and N are related by*

$$\chi_M = \deg(f)\chi_N - R_f,$$

where

$$R_f = \sum_{p\in M} \big(k_p(f) - 1\big)$$

(which is a finite sum because there are only finitely many ramification points).

The genera (or genuses) g_N and g_M are therefore related by

$$g_M = \deg(f)(g_N - 1) + 1 + \frac{R_f}{2}.$$

In particular, it follows that $g_M \geq g_N$.

Elliptic functions[1]

Definition. An *elliptic function* (or *doubly periodic function*) is a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ with two periods ω_1, ω_2 which are linearly independent over $\mathbb{R}$.

In other words, a meromorphic function f on $\mathbb{C}$ is called elliptic if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$ such that $\omega_2/\omega_1 \notin \mathbb{R}$ and $f(z+\omega_k) = f(z)$ for all $z \in \mathbb{C}$, $k = 1, 2$. It follows that

$$f(z+\omega) = f(z) \quad \text{for all} \quad \omega \in \Gamma := \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

Thus, f can be interpreted as a meromorphic function on the Riemann surface $M = \mathbb{C}/\Gamma$, which is a torus.

Theorem 1. *A holomorphic elliptic function is constant.*

In the following lemmas, we consider elliptic functions as function on the torus $M = \mathbb{C}/\Gamma$, and (in Theorem 4) we write $[z]$ for the point $z + \Gamma \in M$.

Theorem 2. *The sum of residues at the poles of an elliptic function is zero.*

Theorem 3. *An elliptic function has as many poles as it has zeros (counting multiplicities).*

Theorem 4. *If the zeros of an elliptic function are $[a_1], \dots, [a_n]$ and its poles are $[b_1], \dots, [b_n]$ (multiple zeros and poles appear multiple times), then*

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \in \Gamma.$$

Constructing elliptic functions: the Weierstrass $\wp$-, ζ-, and σ-functions

The *Weierstrass $\wp$-function* is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma\setminus\{0\}} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big).$$

Proposition. *The above infinite series converges absolutely to a meromorphic function on $\mathbb{C}$ (so $\wp$ is well defined). The convergence is uniform on compact subsets.*

Proposition. *$\wp$ is an even function, i.e., $\wp(z) = \wp(-z)$.*

Proposition. *$\wp$ is doubly periodic with periods ω_1, ω_2.*

(Use the fact that

$$\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$$

is doubly periodic and $\wp$ is even.)

Proposition. *The Laurent expansion of $\wp$ around* 0 *is*

$$\wp(z) = z^{-2} + a_2 z^2 + a_4 z^4 + \dots$$

The Weierstrass $\wp$-function has double poles with zero residue at every lattice point $z \in \Gamma$ and is holomorphic elsewhere.

The *Weierstrass ζ-function* is defined by

$$\zeta(z) = \frac{1}{z} - \int_0^z \Big(\wp(u) - \frac{1}{u^2} \Big)\, du = \frac{1}{z} + \sum_{\omega \in \Gamma\setminus\{0\}} \Big(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \Big),$$

so that

$$\zeta'(z) = -\wp(z)$$

and ζ is odd. The ζ-function is not doubly periodic, but:

[1] This section largely follows chapter 7 in L. Ahlfors, *Complex Analysis*. McGraw-Hill, New York, 1978. Another good source is the book by McKean and Moll, *Elliptic curves. Function theory, geometry, arithmetic.* Cambridge University Press, Cambridge, 1997.

Proposition. *There are constants* $\eta_1, \eta_2 \in \mathbb{C}$ *such that for all* $z \in \mathbb{C}$

$$\zeta(z+\omega_1) = \zeta(z) + \eta_1, \qquad \zeta(z+\omega_2) = \zeta(z) + \eta_2$$

These constants satisfy

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i. \qquad \text{(Legendre relation)}$$

The Weierstrass ζ-function has simple poles with residue 1 at all lattice points $z \in \Gamma$ and is holomorphic elsewhere.

The *Weierstrass σ-function* is defined by

$$\sigma(z) = z\,\exp\left(\int_0^z \left(\zeta(u) - \frac{1}{u}\right) du\right) = z \prod_{\omega\in\Gamma\setminus\{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right),$$

so that

$$\frac{\sigma'}{\sigma} = \zeta$$

and σ is an entire odd function with $\sigma'(0) = 1$. The σ-function is not doubly periodic, but:

Proposition. *For all* $z \in \mathbb{C}$,

$$\sigma(z+\omega_1) = -\sigma(z)\, e^{\eta_1\left(z+\frac{\omega_1}{2}\right)},$$
$$\sigma(z+\omega_2) = -\sigma(z)\, e^{\eta_2\left(z+\frac{\omega_2}{2}\right)}.$$

Using the Weierstrass ζ-function, we can construct elliptic functions with prescribed simple poles and residues.

Theorem 5. *(i) Let $b_1, \dots, b_n \in \mathbb{C}$, $[b_k] \neq [b_m]$ for $k \neq m$, and let $r_1, \dots, r_n \in \mathbb{C}$ with $\sum_k r_k = 0$. Then the function*

$$\phi(z) = \sum_{k=1}^{n} r_k \zeta(z - b_k)$$

is an elliptic function with simple poles at $[b_1], \dots, [b_n]$ and $Res_{b_k}(f) = r_k$.

(ii) Suppose $f(z)$ is an elliptic function with simple poles at $[b_1], \dots, [b_n]$ and $Res_{b_k}(f) = r_k$. Then

$$f(z) = \phi(z) + A$$

for some $A \in \mathbb{C}$.

Similarly, one can construct elliptic functions with higher order poles with given principal parts

$$c_{-m}(z - b_k)^{-m} + \ldots + c_{-1}(z - b_k)^{-1}$$

using the functions $\wp(z)$, $\wp'(z)$, $\wp''(z)$, $\dots$, whose principal parts are z^{-2}, $-2z^{-3}$, $6z^{-4}$, $\dots$.

Using the Weierstrass σ-function, we can construct elliptic functions with prescribed zeros and poles of prescribed order.

Theorem 6. *(i) Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$ satisfying $[a_k] \neq [b_m]$ for all k, m and*

$$\sum_{k=1}^{n} a_k - \sum_{b=1}^{n} b_k = 0. \tag{1}$$

Then

$$\psi(z) = \frac{\sigma(z - a_1)\sigma(z - a_2)\cdots\sigma(z - a_n)}{\sigma(z - b_1)\sigma(z - b_2)\cdots\sigma(z - b_n)}$$

is an elliptic function whose zeros are $[a_1], \dots, [a_n]$ and whose poles are $[b_1], \dots, [b_n]$ (if a number appears multiple times, the zero or pole has the respective order).

(ii) Suppose $g(z)$ is an elliptic function whose zeros are $[a_1], \dots, [a_n]$ and whose poles are $[b_1], \dots, [b_n]$ (multiple zeros and poles appear multiple times). Suppose the representatives a_k and b_k are chosen such that they satisfy (1). (This can be done by Abel's Theorem 4). Then

$$g(z) = B\psi(z)$$

for some $B \in \mathbb{C}^*$.

Meromorphic differentials, integration, and homology

Definition and basic properties

Let R be a Riemann surface with maximal holomorphic atlas $\{z_j : U_j \to \mathbb{C}\}_{j \in I}$. How can we define the derivative of a meromorphic function f on R? For a point $p \in R$, we can choose a chart neighborhood $U_j \ni p$ and consider the meromorphic function $f \circ z_j^{-1}$ on $z_j(U_j) \subseteq \mathbb{C}$. It would be tempting to define the derivative of f at p as the complex number $(f \circ z_j^{-1})'(z_j(p))$. But this number depends on the choice of the chart (U_j, z_j): If (U_k, z_k) is another chart, then $f \circ z_j^{-1} = (f \circ z_k^{-1}) \circ (z_k \circ z_j^{-1})$, so

$$\big(f \circ z_j^{-1}\big)'(z_j(p)) = \big(f \circ z_k^{-1}\big)'(z_k(p)) \cdot \big(z_k \circ z_j^{-1}\big)'(z_j(p)).$$

The derivative of a meromorphic function on R is not a meromorphic function on R, but a meromorphic differential:

Definition. A *meromorphic differential* ϕ on a Riemann surface R is a collection of meromorphic functions ϕ_j on $z_j(U_j) \subseteq \mathbb{C}$, one for each chart (U_j, z_j), such that for all $u \in z_j(U_j \cap U_k)$

$$\phi_j(u) = \phi_k(h_{jk}(u)) \cdot h'_{jk}(u), \tag{2}$$

where $h_{jk} = z_k \circ z_j^{-1}$ is the chart transition map.

If f is a meromorphic function on R, then the collection of functions $\phi_j = (f \circ z_j^{-1})'$ is a meromorphic differential. It is called the *derivative of f* and denoted by df.

Not every meromorphic differential is the derivative of a meromorphic function (unless the Riemann surface is simply connected).

The *sum* $\phi + \psi$ of two meromorphic differentials and the *product* $f\phi$ of a meromorphic function and a meromorphic differential are the meromorphic differentials with local representative functions $\phi_j + \psi_j$ and $(f \circ z^{-1})\phi_j$, respectively. Conversely, if $\psi \neq 0$, then the *quotient* $\frac{\phi}{\psi}$ is a meromorphic function on R. Indeed, since

$$\frac{\phi_j}{\psi_j}(u) = \frac{\phi_k}{\psi_k}\big(h_{jk}(u)\big),$$

there is a unique meromorphic function on R whose restriction to any chart neighborhood U_j is $\frac{\phi_j}{\psi_j} \circ z_j$.

If ϕ is one meromorphic differential, every meromorphic differential equals $f\phi$ for some meromorphic function f.

Remark. The reason why the derivative of a meromorphic function f is not a meromorphic function can now be phrased as follows: The meromorphic functions $\frac{df}{dz_j} = \phi_j(z_j)$, which are defined on chart the neighborhoods U_j, do in general not agree on the overlaps $U_j \cap U_k$.

A meromorphic differential ϕ has a *zero* or *pole* of order n at $p \in R$ if the local representing function ϕ_j has a zero or pole, respectively, of order n at $z_j(p)$ for some (hence every) chart (U_j, z_j) around p. This definition does not depend on the choice of chart because the chart transition maps h_{jk} are biholomorphic and hence their derivatives h'_{jk} have no zeros or poles.

The *residue* of a meromorphic differential ϕ at p is $\operatorname{Res}_p(\phi) := \operatorname{Res}_{z_j(p)} \phi_j$. This does not depend on the choice of chart. (Exercise.) On the other hand, note that the residue of a meromorphic *function* on a Riemann surface is not well defined.

A meromorphic differential without poles is called a *holomorphic differential*.

Examples/Exercises

1. The meromorphic differential dz on $\widehat{\mathbb{C}}$ has no zeros and one double pole at ∞.

2. On a torus $\mathbb{C}/\Gamma$, the function z is not well defined, but the differential dz is. It is a holomorphic differential without zeros.

3. Let R be the elliptic or hyperelliptic Riemann surface

$$w^2 = \textstyle\prod_{k=1}^{N}(z - z_k),$$

where $g \geq 1$ and $N = 2g + 1$ or $N = 2g + 2$. Then

$$\frac{dz}{w}, \frac{z\,dz}{w}, \dots, \frac{z^{g-1}dz}{w}$$

are g holomorphic differentials on R.

Integration of differentials along curves

Let $\gamma : [a,b] \to R$ be a piecewise differentiable curve, which does not go through any poles of the meromorphic differential ϕ. There is a subdivision $a = t_0 < t_1 < \ldots < t_n = b$ such that each $\gamma([t_k, t_{k+1}])$ is contained in some chart neighborhood U_{j_k}, and the *integral of ϕ along γ* is defined as

$$\int_\gamma \phi = \sum_{k=0}^{n-1} \int_{z_{j_k} \circ \gamma|_{[t_k, t_{k+1}]}} \phi_{j_k}(u)\, du.$$

This does not depend on the subdivision, nor on the choice of charts.

The integral is well defined under more general assumptions: The curve may be merely continuous (instead of differentiable) and it may go through poles as long as their residue is zero, and as long as it does not begin or end in a pole. Indeed, we may require the chart neighborhoods U_{j_k} to be simply connected and to contain no poles with nonzero residue. Then each ϕ_{j_k} has an antiderivative F_{j_k} defined in U_{j_k}, and we define

$$\int_\gamma \phi = \sum_{k=0}^{n-1} \big(F_{j_k}(t_{k+1}) - F_{j_k}(t_k)\big).$$

Homology[1]

We will consider finite formal sums with coefficients in $\mathbb{Z}$.

A 0-*chain* is a formal sum $c_0 = \sum_{k=1}^n n_k p_k$ of points $p_k \in R$. The $\mathbb{Z}$-module (or, which is the same, the Abelian group) of 0-chains in R is denoted by $C_0(R)$.

A 1-*chain* is a formal sum

$$c_1 = \sum_{k=1}^n n_k \gamma_k$$

of continuous maps $\gamma_k : [0,1] \to R$. The $\mathbb{Z}$-module of 1-chains in R is denoted by $C_1(R)$.

The *boundary* of a 1-chain c_1 is the 0-chain

$$\partial c_1 = \sum_{k=1}^n n_k \big(\gamma_k(1) - \gamma_k(0)\big).$$

The boundary map $\partial : C_1(R) \to C_0(R)$ is $\mathbb{Z}$-linear (or, which is the same, a homomorphism of Abelian groups). A 1-chain c with $\partial c = 0$ is called *closed*, or a *cycle*.

The *integral of a meromorphic differential ϕ on R over a 1-chain* is defined by

$$\int_{c_1} \phi = \sum_{k=1}^n n_k \int_{\gamma_k} \phi.$$

This is well-defined if the curves γ_k do not end in poles or pass through poles with nonzeoro residue.

If we define the *integral of a meromorphic function over a 0 chain* by

$$\int_{c_0} f = \sum_{k=1}^n n_k f(p_k),$$

then

$$\int_{c_1} df = \int_{\partial c_1} f.$$

In particular, if c_1 is closed, then $\int_{c_1} df = 0$.

[1] There are in fact many different homology theories. The particular concept of homology that we are concerned with here is called "singular homology with integer coefficients".

Theorem. *If a meromorphic differntial ϕ has only poles with zero residue and $\int_c \phi = 0$ for all closed 1-chains c (that do not start or end in a pole), then $\Phi = df$ for a meromorphic function $f(p) = \int_{p_0}^{p} \phi$.*

A 2-*chain* is a formal sum

$$c_2 = \sum_{k=1}^{n} n_k \tau_k$$

of continuous maps $\tau_k : \Delta \to R$, where Δ is the triangle

$$\Delta = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 0\}.$$

(Any other triangle would do equally well.) The $\mathbb{Z}$-module of 2-chains in R is denoted by $C_2(R)$.

The *boundary* of a 2-chain c_2 is the 1-chain

$$\partial c_2 = \sum_{k=1}^{n} n_k \, \partial \tau_k,$$

where

$$\partial \tau = \big(t \mapsto \tau(t,0)\big) - \big(\tau \mapsto \tau(0,t)\big) + \big(t \mapsto \tau(1-t,t)\big).$$

The boundary map $\partial : C_2(R) \mapsto C_1(R)$ is $\mathbb{Z}$-linear. (This map is denoted by the same symbol ∂ as the boundary map for 1-chains.)

Note that $\partial\tau$ is a closed 1-chain. By linearity, the boundary of any 2-chain is a closed 1-chain, i.e.,

$$\partial \circ \partial = 0.$$

The converse is not true in general: Not all closed 1-chains are boundaries.

A closed 1-chain c_1 is called *null-homologous* if it is a boundary, i.e., if there is a 2-chain c_2 with $\partial c_2 = c_1$. Two 1-chains, c_1 and $\tilde{c}_1$ are called *homologous* if $c_1 - \tilde{c}_1$ is null-homologous.

Some important examples, as exercise:

- *Reversal of orientation*: If $\gamma_2(t) = \gamma_1(1-t)$, then the chain $\gamma_1 + \gamma_2$ is null-homologous, i.e., γ_2 is homologous to $-\gamma_1$.
- *Concatenation*: If $\gamma_1(1) = \gamma_2(0)$ and γ_3 is the concatenation of γ_1 and γ_2, then γ_3 is homologous to $\gamma_1 + \gamma_2$.
- If two curves are homotopic with fixed endpoints, then they are homologous.
- If two closed curves are freely homotopic, then they are homologous.

Remark. If a and b are two curves starting and ending in the same point, and a^{-1}, b^{-1} denote the curves with reversed orientation, then the concatenation $aba^{-1}b^{-1}$ is null-homologous but need not be null-homotopic. In fact, the first homology group is the abelianization of the fundamental group. (This is a special case of Hurewicz theorem.)

Theorem (Cauchy's theorem for Riemann surfaces)**.** *If a meromorphic differential ϕ has no poles with nonzero residue and c is a null-homologous 1-chain, then $\int_c \phi = 0$.*

Corollary. *Under the same assumptions, if c and $\tilde{c}$ are homologous 1-chains, then $\int_c \phi = \int_{\tilde{c}} \phi$.*

Recall that a Riemann surface of genus g is homeomorphic to a $4g$-gon P_g with sides glued according to the scheme

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}. \tag{1}$$

Suppose one such homeomorphism is chosen, so that we can consider the $a_1, \dots, a_b$ and $b_1, \dots, b_g$ as curves in R.

Theorem 7. *Every closed* 1*-chain c_1 is homologous to an integer linear combination of $a_1, \dots, a_b$ and $b_1, \dots, b_g$:*

$$c_1 \sim \sum_{k=1}^{g} n_k a_k + \sum_{k=1}^{g} n_{g+k} b_k.$$

In fact, a stronger statement is true: The cycles a_k, b_k (or rather their homology classes) form a basis of the first homology group, i.e., the coefficients n_k are uniquely determined. A homology basis (like a_k, b_k) that comes like from a representation of R as $4g$-gon P_g with sides glued accroding to the scheme (1) is called a *canonical homology basis*.

Abelian differntials

Proposition. *The sum of residues of a meromorphic differential on a compact Riemann surface is zero:* $\sum_p \operatorname{Res}_p(\omega) = 0$.

Definition. A meromorphic differential on a compact Riemann surface is also called an *Abelian differential*

- *of the first kind* if it is holomorphic,
- *of the second kind* if it has poles but zero residue everywhere,
- *of the third kind* if it has poles with nonzero residue.

Let R be a compact Riemann surface of genus g, and let $a_1, \dots, a_g$, $b_1, \dots, b_g$ be a canonical homology basis.

The *A-periods* and *B-periods* of an Abelian differential ω are the numbers

$$\int_{a_j} \omega \quad \text{and} \quad \int_{b_j} \omega, \quad \text{for} \quad j = 1, \dots, g.$$

We assume the curves a_j, b_j are chosen so that they do not go through any pole with nonzero residue.

Proposition. *Suppose ω is an Abelian differential of the second kind with vanishing A- and B-periods: $\int_{a_j} \omega = \int_{b_j} \omega = 0$ for $j = 1, \dots, g$. Then ω is the derivative of a meromorphic function $f(p) = \int_{p_0}^{p} \omega + c$.*

Theorem (Uniqueness)**.** *If ω is a holomorphic differential with vanishing A-periods, i.e.,*

$$\int_{a_1} \omega = \int_{a_2} \omega = \dots = \int_{a_g} \omega = 0,$$

then $\omega = 0$.

The proof uses Stokes' theorem and the calculus of differential forms to show that

$$\|\omega\|_{L^2}^2 = \tfrac{i}{2} \int_R \omega \wedge \bar{\omega} = \tfrac{i}{2} \textstyle\sum_{j=1}^{g} (A_j \bar{B}_j - B_j \bar{A}_j). \tag{2}$$

Corollary. *An Abelian differential of the first kind is uniquely determined by its A-periods.*

Theorem (Existence I)**.** *There exist Abelian differentials of the first kind $\omega_1, \dots, \omega_g$ with $\int_{a_j} \omega_k = \delta_{jk}$.*

Corollary (of Uniqueness and Existence I)**.** *The complex vector space $\Omega^1(R)$ of holomorphic differentials on R is g-dimensional, and the linear map*

$$\Omega^1(R) \to \mathbb{C}^g, \qquad \omega \mapsto \Big(\int_{a_1} \omega, \int_{a_2} \omega, \dots, \int_{a_g} \omega \Big)$$

is an isomorphism.

The differntials $\omega_1, \dots, \omega_g$ are uniquely determined. They form a *canonical basis* of $\Omega^1(R)$.

Theorem (Existence II)**.** *For a point $p \in R$, a coordinate z around p with $z(p) = 0$, and an integer $n \geq 2$, there exists a Abelian differential of the second kind $\omega^{II}_{p,n}$ such that near p*

$$\omega^{II}_{p,n} = \big(z^{-n} + O(1)\big)\, dz,$$

$\omega^{II}_{p,n}$ has no other poles, and all A-periods of $\omega^{II}_{p,n}$ vanish.

Theorem (Existence III)**.** *For two points $p, q \in \mathbb{R}$, there exists an Abelian differential of the third kind $\omega^{III}_{p,q}$ that has simple poles with residue $+1$ and -1 at p and q, respectively, no other poles, and vanishing A-periods.*

The Abelian differentials $\omega^{II}_{p,n}$, and $\omega^{III}_{p,q}$ are also uniquely determined. Note, however, that the differentials $\omega^{II}_{p,n}$ depend not only on p and n but also on the choice of coordinate z around p.

Abelian differentials of the second and third kind with vanishing A-periods are called *normalized*.

Theorem (Existence and Uniqueness Summary)**.** *For the following data:*

- *N points $p_1, \dots, p_N \in R$,*
- *local coordinates $z_1, \dots, z_N$ around these points with $z_k(p_k) = 0$,*
- *complex numbers $a^{(k)}_{-1}, a^{(k)}_{-2}, \dots, a^{(k)}_{-n_k}$ for $k = 1, \dots, N$ satisfying $\sum_{k=1}^{N} a^{(k)}_{-1} = 0$,*
- *complex numbers $A_1, \dots, A_g$,*

there exists a unique Abelian differential with the series expansions

$$\omega = \big(a^{(k)}_{-n_k} z_k^{-n_k} + \dots + a^{(k)}_{-1} z_k^{-1} + O(1)\big)\, dz_k$$

near the points p_k, no poles elsewhere, and the A-periods $\int_{a_j} \omega = A_j$.

Abel's Theorem

Jacobi variety and Abel map

Let $\omega_1, \dots, \omega_g$ be a canonical homology basis, so the A-periods are $\int_{a_k} \omega_j = \delta_{jk}$, and let $B = (B_{jk})$ be the matrix of B-periods:

$$B_{jk} = \int_{b_k} \omega_j.$$

For $p_0, p_1 \in R$, the integral $\int_{p_0}^{p_1} \omega_j$ is not uniquely determined, because it depends on the path γ from p_0 to p_1. Since the difference of two such paths $\gamma, \tilde{\gamma}$ is a cycle, Theorem 7 says $\tilde{\gamma} - \gamma \sim \sum_{k=1}^{g} n_k a_k + \sum_{k=1}^{g} m_k b_k$ for $n, m \in \mathbb{Z}^g$, and hence

$$\int_{\tilde{\gamma}} \omega_j - \int_{\gamma} \omega_j = n_j + \sum_k B_{jk} m_k.$$

Definition. The *Jacobi variety* of R is $\mathrm{Jac}(R) = \mathbb{C}^g/\Gamma$, where $\Gamma = \mathbb{Z}^g + B\,\mathbb{Z}^g$. The *Abel map* is

$$\mathscr{A} : R \to \mathrm{Jac}, \qquad \mathscr{A}(p) = \Big(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2, \dots, \int_{p_0}^{p} \omega_g \Big)$$

for an arbitrary but fixed point $p_0 \in R$.

Theorem. *The matrix of B-periods is symmetric, $B_{jk} = B_{kj}$, and its imaginary part $\operatorname{Im} B$ is positive definite.*

Symmetry follows from the following Lemma, by taking ω_j and ω_k as ω and ω'.

Lemma. *Let ω and ω' be two holomorphic differentials with A- and B-periods A_k, B_k and A'_k, B'_k, respectively. Then $\sum_{k=1}^{g} A_k B'_k - B_k A'_k = 0$.*

Remark. This Lemma is a special case of *Riemann's bilinear relations*, which hold more generally for closed but not necessarily holomorphic differentials. These general relations imply equation (2), which in turn implies positive definiteness of B by taking $\omega = \sum_{j=1}^{g} a_j \omega_j$.

Corollary. *In particular, $\operatorname{Im} B$ is nonsingular. Together with the standard unit vectors $e_1, \ldots, e_g \in \mathbb{C}^g$, the columns of B form a real basis of the $2g$-dimensional real vector space $\mathbb{R}^{2g} = \mathbb{C}^g$. Γ is a $2g$-dimensional lattice in $\mathbb{R}^{2g}$. $\operatorname{Jac}(R)$ is a torus of real dimension $2g$.*

In fact, the Abel map is an immersion, i.e., the derivative $d\mathscr{A}$ is nowhere zero (this follows from the Riemann-Roch Theorem) and injective (this follows from Abel's theorem).

In particular, for $g = 1$, the Abel map is a biholomorphic map $R \to \mathbb{C}/\Gamma$.

Divisors and Abel's theorem

A *divisor* on R is a formal sum of points in R.[1] The *divisor of a nonzero meromorphic function f* on R is

$$(f) = \sum_{k=1}^{M} m_k p_k - \sum_{k=1}^{N} n_k q_k,$$

where $p_1, \ldots, p_M$ are the zeros of f and $m_1, \ldots, m_M$ are their orders, and $q_1, \ldots, q_N$ are the poles of f and $n_1, \ldots, n_N$ are their orders.

A divisor D on R is called a *principal divisor* [in German: *Hauptdivisor*] if $D = (f)$ for some meromorphic function f.

The *degree* of a divisor $D = \sum_k n_k p_k$ is defined as

$$\deg(D) = \sum_k n_k.$$

Thus, the degree of any principal divisor is zero,

$$\deg\big((f)\big) = 0,$$

because a meromorphic function has as many zeros as poles, counting multiplicity. (Note that the degree of a divisor and the degree of a branched covering, see lecture 5, are different concepts.)

For a divisor $D = \sum_k n_k p_k$ we define the Abel map as

$$\mathscr{A}(D) = \sum_k n_k \mathscr{A}(p_k) = \sum_k n_k \int_{p_0}^{p_k} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}.$$

If $\deg D = 0$, then $D = p_1 + \ldots + p_n - q_1 - \ldots - q_n$ (where the p_k and q_k need not all be distinct) and $\mathscr{A}(D)$ does not depend on the choice of p_0:

$$\mathscr{A}(D) = \int_{q_1}^{p_1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} + \ldots + \int_{q_n}^{p_n} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix}.$$

Theorem (Abel)**.** *A divisor D is a principal divisor if and only if $\deg(D) = 0$ and $\mathscr{A}(D) = 0$.*

Corollary. *The Abel map $\mathscr{A} : R \to \operatorname{Jac}(R)$ is injective.*

Proof. Suppose on the contrary that $\mathscr{A}(p_1) = \mathscr{A}(p_2)$ for $p_1 \neq p_2$. Then $\mathscr{A}(p_1 - p_2) = 0$ and by the theorem there is a meromorphic function f on R with only one simple pole at p_1. This contradicts the sum of residues being zero. ☐

[1] A divisor is therefore the same as a 0-chain with coefficients in $\mathbb{Z}$. They are just called differently in different contexts.

Theorem of Riemann and Roch[1]

The *divisor of a nonzero meromorphic differential* is defined in the same way as the divisor of a nonzero meromorphic function: $(\omega) = \sum_k m_k p_k - \sum n_k q_k$, where p_k and q_k are the zeros and poles of ω, respectively, and m_k and n_k are their orders.

A divisor D is called a *canonical divisor* if $D = (\omega)$ for some Abelian differential.

- $(fg) = (f) + (g)$
- $(f\omega) = (f) + (\omega)$
- The difference of two canonical divisors is a principal divisor.
- All canonical divisors have the same degree.

A partial order on the set of divisors is defined by

$$D \geq 0 \quad :\Longleftrightarrow \quad D = \sum_k n_k p_k \text{ with } n_k \geq 0,$$
$$D \geq D' \quad :\Longleftrightarrow \quad D - D' \geq 0.$$

For any divisor D a subspace $L(D)$ of the complex vector space of meromorphic functions is defined by

$$L(D) = \{f \text{ meromorphic function on } R \mid (f) \geq -D \text{ or } f = 0\},$$

If $D = \sum m_k p_k - \sum n_k q_k$ for positive m_k, n_k, then $L(D)$ consists of the meromorphic functions that have only poles of order at most m_k at p_k and zeros of order at least n_k at q_k.

Equally, a subspace $H(D)$ of the space of meromorphic differentials is defined by

$$H(D) = \{\omega \text{ Abelian differential on } R \mid (\omega) \geq -D \text{ or } \omega = 0\}.$$

We denote the dimensions of these subspaces by

$$l(D) = \dim L(D), \quad h(D) = \dim H(D).$$

- If D and D' differ by a principal divisor, i.e., $D' = D + (f)$, then $l(D) = l(D')$ and $h(D) = h(D')$.
- If K is any canonical divisor, i.e., $K = (\omega)$, then

$$h(D) = l(K + D).$$

Theorem (Riemann–Roch). *For any divisor D,*

$$l(D) = \deg(D) + 1 - g + h(-D).$$

Proof only for divisors of the form $D = \sum_{k=1}^d p_k$ for distinct p_k. Uses the following Lemma:

Lemma. *Let $p_1, \dots, p_d \in R$ be distinct points, let $z_1, \dots, z_d$ be local coordinates with $z_k(p_k) = 0$, and let $\tilde{\omega} = \sum_{k=1}^d x_k \omega^{II}_{p_k,2}$ for $x \in \mathbb{C}^k$, so that $\tilde{\omega}$ has vanishing A-periods and double poles at p_k with asymptotics*

$$\tilde{\omega} = \big(x_k z_k^{-2} + O(1)\big)\, dz_k, \quad \text{for } p \to p_k.$$

Further, let $\omega_1, \dots, \omega_g$ be the basis of holomorphic differentials with $\int_{a_k} \omega_j = \delta_{jk}$, and suppose

$$\omega_j = \big(c_{jk} + O(z_k)\big)\, dz_k, \quad \text{for } p \to p_k.$$

Then

$$\int_{b_j} \tilde{\omega} = 2\pi i \sum_{k=1}^d c_{jk} x_k.$$

Corollaries. *(i) If $deg(D) \geq g + 1$, then there exists a nonconstant meromorphic function in $L(D)$.*

(ii) Any Riemann surface of genus 0 is conformally equivalent with $\widehat{\mathbb{C}}$.

(iii) There is no point where all holomorphic differentials vanish simultaneously. This implies that the Abel map is an immersion.

(iv) The degree of a canonical divisor is $2g - 2$.

(v) Every Riemann surface of genus 2 is conformally equivalent with a hyperelliptic curve.

[1] This is bonus material. For lack of time, I could not cover this in the lecture.