Quadrilateral sets and liftings

In this chapter we will investigate an important concept of projective geometry: **quadrilateral sets**. On the one hand these configurations can be considered as a generalization of harmonic points. On the other hand they have a close relation to the liftability of lower dimensional point configurations to prescribed higher dimensional scenarios.

### 8.1 Points on a line

We will start our studies with the algebraic consequences of having \( n \) points on a line. Any bracket formed by three collinear points will automatically be zero. Via Grassmann-Plücker-Relations this vanishing bracket causes other relations among the other (non-zero) brackets. (From now on, we will freely omit the commas in brackets whenever no confusion can arise to make the formulas a bit more compact and readable, thus we may write \([abc]\) instead of \([a,b,c]\)). Consider the three-summand Grassmann-Plücker-Relation

\[
[abc][axy] - [abx][acy] - [aby][acx] = 0.
\]

We know that this equation holds for arbitrary points \( a, b, c, x, y \) of \( \mathbb{R}P^2 \). If in a configuration in addition we know that \( a, b, c \) are collinear, then the first summand of this equation vanishes and we obtain the equation:

\[
[abx][acy] = [aby][acx].
\]

This relation generalizes to more general contexts:

**Theorem 8.1.** Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}P^2 \) be \( 2n \) collinear points (not necessarily distinct), and let \( x_1, \ldots, x_n \in \mathbb{R}P^2 \) be \( n \) arbitrary additional points. Then the following bracket equation holds for any permutation \( \pi \in S_n \):

\[
\prod_i [a_i, b_i, x_i] = \prod_i [a_i, b_i, x_{\pi(i)}].
\]
Proof. In principle, one can proof this result by a suitable linear combination of Grassmann-Plücker-Relations. However, we will use this result to introduce the technique of “proof by specialization”. The argument goes as follows. The conclusion of the proof is obviously a projective invariant property (every letter occurs as often on the left as on the right). Thus it is invariant under projective transformations and rescaling of homogeneous coordinates. Thus we may w.l.o.g. assume that all the last entries of the homogeneous coordinates are 1. In this case the determinant \( [a, b, c] \) equals twice the oriented area of the corresponding triangle. For this situation we can provide a very elementary proof.

The area of the triangle \((a_i, b_i, x_i)\) can be calculated as \( |a_i, b_i| \cdot h(x_i)/2 \), where \( |a_i, b_i| \) is the oriented distance from \( a_i \) to \( b_i \) and \( h(x_i) \) is the altitude of point \( x \) over the line on which the \( a_i \) and \( b_i \) lie. Thus both sides of the expression \( \prod_i [a_i, b_i, x_i] = \prod_i [a_i, b_i, x_{\pi(i)}] \) must be equal, since they only represent two different ways of ordering the factors of a product.

\( \square \)

8.2 Quadrilateral sets

Our next example studies the question under which conditions points on a line are the projection of a certain incidence configuration in \( \mathbb{RP}^3 \). Consider the picture in Figure 8.2. The four distinct blue lines intersect in six points. These six points are projected (with eye-point \( o \)) to the black line \( \ell \). How can we characterize whether six points on \( \ell \) arise from such a projection. We will present several approaches to this question.

First of all, since all six points are collinear the characterization must be expressible as a one-dimensional condition on the line. This condition in turn must be expressible purely on the level of determinant equations. Since the points are projected with projection center \( o \) it must be possible to express
the condition as a determinant expression in which each determinant involves point $o$. For deriving this equation we will now introduce a technique that is also applicable in many other contexts.

We consider the four collinearities $[abc] = [aes] = [bdf] = [cde] = 0$ that hold in our picture. From these collinearities we obtain the following five equations:

$$
\begin{align*}
[abc] = 0 & \Rightarrow [abc][bcf][cad] = [abf][bcf][cae] \\
[aes] = 0 & \Rightarrow \underline{[aeo][afb]} = \underline{[aeb][afa]} \\
[bfd] = 0 & \Rightarrow \underline{[bfo][bdc]} = \underline{[bff][bdo]} \\
[cde] = 0 & \Rightarrow \underline{[cdo][cea]} = \underline{[cda][ceo]}
\end{align*}
$$

The last three are direct consequences of Grassmann-Plücker-Relations. The first equation is an application of Theorem 8.1. Multiplying all left sides and all right sides, and canceling determinants that appear on both sides (those that are not underlined) we arrive at the equation:

$$
[aeo][bfo][cdo] = [afq][bdo][ceo].
$$

This is the desired characterization. Since in each bracket the point $o$ is involved, we can also read this expression as a rank 2 expression of the corresponding projections on the line.

Before we study the symmetry of the bracket expression we will have a look at two other ways of deriving this formula.

Consider the picture in Figure 8.3 left. There the point of projection has been moved to infinity. We want to give an affine argument from which we can generate the desired formula. This time we will directly head for the rank 2 formula: Under which conditions are six points $a, b, c, d, e, f$ on a line
liftable to the blue incidence configuration. We want a non-trivial lifting, in which not all lines are identical. We furthermore assume that points whose lifted images should be collinear do not coincide on the line. We assume that the points have homogeneous (rank 2) coordinates \((x_a, 1), \ldots, (x_f, 1)\). A lifting of the six points corresponds to an assignment of altitudes \(h_a, \ldots, h_f\) to each of the points. Assume that we have such a lifting such that the lifted points \(a, b, c\) are collinear. The lifted points have homogeneous coordinates 
\[
(x_a, 1, h_a), (x_b, 1, h_b), (x_c, 1, h_c).
\]
Their collinearity is expressed as
\[
0 = \det \begin{pmatrix} x_a & 1 & h_a \\ x_b & 1 & h_b \\ x_c & 1 & h_c \end{pmatrix} = [ab]h_c - [ac]h_b + [bc]h_a.
\]

We get similar expressions for the other four lines. In a lifting these four conditions have to be satisfied simultaneously. Thus any lifting corresponds to a solution of the linear system of equations:
\[
\begin{pmatrix}
+[bc] -[ac] +[ab] & 0 & 0 & 0 \\
+[ef] & 0 & 0 & 0 \\
0 & +[df] & 0 & -[bf] +[ae] \\
0 & 0 & +[de] -[ce] +[cd] & 0
\end{pmatrix}
\begin{pmatrix} h_a \\ h_b \\ h_c \\ h_d \\ h_e \\ h_f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

In order to get non-trivial liftings the solution space of this system has to have at least dimension three (having all lifted points in trivial position on a single line already accounts for two dimensions). Thus if the points are liftable, the the matrix must have rank at most 3. Thus any \(4 \times 4\) subdeterminant must vanish. Considering the last four columns we get:
\[
\det \begin{pmatrix}
+[ab] & 0 & 0 & 0 \\
0 & 0 & -[af] +[ae] \\
0 & -[bf] & 0 & +[bd] \\
+[de] -[ce] +[cd] & 0
\end{pmatrix} = [ab] \cdot ([ae][bf][cd] - [af][bd][ce])
\]

Since \([ab]\) was assumed to be non-zero, the vanishing of this determinant implies our desired characterization.
\[
[ac][bf][cd] = [af][bd][ce].
\]

Points on a line that satisfy this relation are called a quadrilateral set. By considering other columns of our matrix we could derive at similar looking equivalent bracket expressions.

Here is another way of deriving the same formula from a different geometric situation formula. The right picture of Figure 8.3 shows the dual situation of the one we considered so far. We draw the six lines through four points \(x, z, p, q\).
8.3 Symmetry and generalizations of quadrilateral sets

The quadrilateral set configuration has interesting inner structures. In our labeling there are three pairs of points \((a, d), (b, e)\) and \((c, f)\) such that the points of each pair do not share a line of the configuration. Every line of the complete quadrilateral is obtained by selecting exactly one point from each of the pairs. For instance the line \((a, b, c)\) takes the first point of each of the pairs. In our expression

\[
\begin{align*}
[ae][bf][cd] &= [ce][af][bd].
\end{align*}
\]

the line \((a, b, c)\) plays a special role. The two monomials of the expression are formed by three brackets each of these brackets contains exactly one point of the line and one other point such that they do not form one of the three pairs above. For each line we get exactly one such characterization of the quadrilateral set.

Besides the four lines in our original quadrilateral there are also four other lines that can be formed by taking exactly one point from each of the
Fig. 8.4. Incidence theorems from quadrilateral sets

pairs. These lines \((d, e, f), (a, b, f), (b, c, d)\) and \((a, c, e)\) describe the associated complete quadrilateral to our original one. The symmetry of \([ae][bf][cd] = [ce][af][bd]\) also implies that this expression is as well a characterization of the quadrilateral set generated by the associated complete quadrilateral. The situation is illustrated in Figure 8.4 left. The blue part of the picture is our original quadrilateral set configuration, the green part is the complementary one. In particular if we interchange the role of two points in one of the pairs \((a, d), (b, e), (c, f)\) then we transfer the complete quadrilateral into its associated one. This implies that the triple of pairs characterizes the quadrilateral set. The right part of Figure 8.4 is an illustration of obtaining a quadrilateral set by projection or by intersection. Both pictures represent projective incidence theorems. If all coincidences except the last one are satisfied then the last last one is satisfied automatically.

Finally, we want to explore how the notion of quadrilateral sets can be generalized. One way of doing this is to use one of the lines of the complete quadrilateral itself for the projection. Figure 8.5 left shows the situation for the usual quadrilateral set. The six points on the base line are the three projections \(a_1, a_2, a_3\) of the points of a triangle and the intersections \(b_1, b_2, b_3\) with its sides. We get the equation

\[[a_1, b_1][a_2, b_2][a_3, b_3] = [a_1, b_3][a_2, b_1][a_3, b_2].\]

If we consider an arbitrary \(n\)-gon with vertices \(1, \ldots, n\), the projections of its vertices \(a_1, \ldots, a_n\) to a line \(\ell\) and the intersections \(b_i = \ell \wedge (i \vee (i + 1))\) (indices modulo \(n\)), then we get the equation:

\[\prod_i [a_i, b_i] = \prod_i [a_i, b_{i-1}]\]
This formula can be proved easily by the techniques we used for the characterization of quadrilateral sets. A non-trivial lifting of the segment \((i, i + 1)\) implies the existence of non-zero altitudes \(h_i, h_{i+1}\) such that

\[
\frac{[a_i, b_i]}{[a_{i+1}, b_i]} = \frac{h_i}{h_{i+1}}.
\]

Forming the product over all \(i\) yields (indices modulo \(n\)):

\[
\prod_{i=1}^{n} \frac{[a_i, b_i]}{[a_{i+1}, b_i]} = \prod_{i=1}^{n} \frac{h_i}{h_{i+1}} = 1.
\]

Which is equivalent to the desired formula.

8.4 Quadrilateral sets and von Staudt

Let us reconsider the original von Staudt constructions we got to know in Section 5.6. The von Staudt constructions provided a tool for performing addition and multiplication with respect to a projective basis \(0, 1, \infty\) on a line. Figure 8.6 shows the situation with the point \(\infty\) moved to a finite position (compare the drawing with Figure 5.5). We observe that the relevant points of the calculation are the intersection with the six lines through for other points. In other words we see that the von Staudt constructions are based on a quadrilateral set construction.

What makes von Staudt construction work is the fact that (in the usual identification of the projective line with \(\mathbb{R} \cup \{\infty\}\)) the following triples of (ordered) pairs define quadrilateral sets:

\[
((0, x + y); (x, y); (\infty, \infty)) \quad \text{and} \quad ((0, 0); (x, y); (1, x \cdot y)).
\]

The reader is invited to check this fact by hand-calculation explicitly.
8.5 Slope conditions

We continue to play with quadrilateral sets. What happens, for instance, if we intersect the six lines through four (finite) points with the line at infinity? In this case we get a quadrilateral set on the line at infinity. Form a projective point of view this is not at all a special case. However, we want to interpret the quadrilateral set from the perspective of a special coordinatization. With respect to the usual standard embedding of the Euclidean plane at the affine $z = 1$ plane infinite points have coordinates of the form $(x, y, 0)$. A finite line $\ell$ with equation $ax + by + c = 0$ intersects the line at infinity in the point $(b, -a, 0)$. We could rewrite the line equation as $y = -\frac{a}{b}x - \frac{c}{b}$. Thus the intersection point has (after rescaling of homogeneous coordinates) the coordinates $(1, s, 0)$, where $s$ is the slope of the line. If we choose the following basis for the line at infinity $0 = (1, 0, 0)$, $\infty = (0, 1, 0)$ and $1 = (1, 1, 0)$. The parameter of a point with respect to this basis is exactly the slope of the line bundle passing through it.

Taking these considerations into account we can, after four finite points are given, read off a quadrilateral set condition for the slopes of six lines spanned.

![Fig. 8.6. Geometric addition and multiplication, with a finite point $\infty$](image)

![Fig. 8.7. Line slopes between four points](image)
by them. Figure 8.7 illustrates this fact. If \( a, \ldots, f \) are the slopes of the three lines the quadrilateral set condition reads:

\[
(a - e) \cdot (b - f) \cdot (c - d) = (a - f) \cdot (b - d) \cdot (c - e).
\]

In the concrete example we get:

\[
\left(\frac{1}{3} - (-\frac{1}{7})\right) \cdot (-5 - 3) \cdot (-\frac{1}{2} - (-\frac{7}{5})) = \left(\frac{1}{3} - 3\right) \cdot (-5 - (-\frac{7}{5})) \cdot (-\frac{1}{2} - (-\frac{7}{5}))
\]

which is reduced to the identity

\[
\frac{10}{21} \cdot (-8) = -\frac{8}{3} \cdot \frac{-18}{5} \cdot \left(-\frac{5}{14}\right).
\]

The relation of slopes and quadrilateral sets can be used to derive interesting theorems in affine geometry (in which parallels can be used as a primitive predicate). In Figure 8.4 (left) we illustrated the fact that quadrilateral sets arise in two combinatorially different ways as projection of the intersection of
four lines. This translates to the fact illustrated in Figure 8.8. In this picture lines with identical colors are parallel. If four points in the plane are given the slopes of the six lines spanned by them for a quadrilateral set. Thus we can find lines which are parallel to these lines that form a combinatorially different drawing of the slopes between four points. The corresponding lines that pass through a point in the left picture form a triangle in the right picture.

Combining our considerations on slopes and the relation of quadrilateral sets with von Staudt constructions we can also perform geometric addition and multiplication on the level of slopes. Figure 8.9 shows the two corresponding configurations. After fixing lines with slopes 0, and $\infty$ the left drawing demonstrates how to perform addition of two slopes $x$ and $y$. In the right drawing furthermore a line with slope 1 is fixed. The construction forces multiplication of the slopes $x$ and $y$. In both cases it is very easy to prove the relations of slopes by elementary considerations.

As an example of combining several slope addition and multiplication devices the drawing in Figure 8.10 shows a configuration in which after fixing the slopes 0, 1 and $\infty$ some of the lines are forced to have slope $\sqrt{2}$. One subconfiguration is used to perform the operation $1 + 1 = 2$ and another configuration is used to calculate $x \cdot x = 2$. The slope $x$ is then forced to be either $\sqrt{2}$ or $-\sqrt{2}$.

8.6 Involutions and quadrilateral sets

There is also a very interesting connection of quadrilateral sets to projective transformations on a line. For this we have to consider projective involutions
8.6 Involutions and quadrilateral sets

The defining property of an involution is that \( \tau(\tau(p)) = p \) for every point \( p \). Every projective involution on \( \mathbb{RP}^1 \) can be expressed by multiplication with a \( 2 \times 2 \) matrix \( T \) that satisfies \( T^2 = \lambda \cdot \text{Id} \). Involutions are closely related to geometric reflections, since the characterizing property of a reflection is that the mirror image of the mirror image is the original again.

We now get:

**Theorem 8.2.** Let \( \tau: \mathbb{RP}^1 \to \mathbb{RP}^1 \) be a projective involution that is not the identity and let \( a, b, c \) arbitrary points in \( \mathbb{RP}^1 \), then the pairs

\[(a, \tau(a)), (b, \tau(b)), (c, \tau(c))\]

form a quadrilateral set.

**Proof.** Since \( T \) is an involution we have \( T^2 = \lambda \cdot \text{Id} \). This implies \( \det(T)^2 = \det(T^2) = \det((\lambda \cdot \text{Id})^2) = \lambda^2 \). Hence \( \det(T) \) is either \( +\lambda \) or \( -\lambda \). We will first exclude the case \( \det(T) = \lambda \). For this the fact that \( T \) is a \( 2 \times 2 \) matrix is crucial. Let \( T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \). We then get

\[T^2 = \begin{pmatrix} \alpha^2 + \beta \gamma & \alpha \beta + \beta \delta \\ \alpha \gamma + \beta \gamma & \gamma \beta + \delta^2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} . \]

This can only be satisfied if either \( \alpha = -\delta \) or if \( \gamma = \beta = 0 \). The first case leads to \( \det(T) = -\lambda \) (check it). The second case implies \( \alpha^2 = \delta^2 \). Since again \( \tau \) was assumed not to be the identity this case implies \( \alpha = -\delta \) which again implies (together with \( \beta = \gamma = 0 \)) the equation \( \det(T) = -\lambda \). We now consider the monomial

\([a, Tb][b, Tc][c, Ta] \).

Applying the transformation \( T \) to each of the points of this monomial transfers this monomial to

\([Ta, T^2b][Tb, T^2c][Tc, T^2a] = [Ta, \lambda b][Tb, \lambda c][Tc, \lambda a] = -\lambda^3[b, Tb][c, Tb][a, Ta] \).
On the other hand the transferred monomial must satisfy the equation
\[ [Ta, T^2b][Tb, T^2c][Tc, T^2a] = (\text{det}(T))^3[a, Tb][b, Tc][c, Ta]. \]
Comparing this two terms and using the fact \( \text{det}(T) = -\lambda \) we get
\[ [a, Tb][b, Tc][c, Ta] = [a, Tc][b, Ta][c, Tb] \]
This is exactly the characterization of a quadrilateral set. \( \square \)

Let us use this fact to derive immediate conclusion about configurations that concern the slopes of lines. For this we consider involutions on the line at infinity. We will consider two natural types of involution that are related to operations in the Euclidean plane. The first one is a rotation about 90 around an arbitrary point. Such a rotation is clearly an involution. It indices an involution on the line at infinity. Slopes are transferred by such a rotation to perpendicular slopes. For our incidence theorem we start with three arbitrary line slopes (compare Figure 8.11). In the picture they are black yellow and green. Then we construct line slopes that are perpendicular to these slopes. By the above theorem they the six slopes form a quadrilateral set. Thus one can draw a projection of a tetrahedron with exactly these line slopes. Lines that do not share a point in the tetrahedron are perpendicular to each other. Looking at the right part of Figure 8.11 one observes that this statement is nothing else than a strange way to derive a well known result: the altitudes in a triangle meet in a point.

There is another theorem that is not known so well that can be derived by the same argument. For this we consider an involution that arises from a line reflection. Figure 19.1 on the left shows six lines that are pairwise in a mirror relation to each other. The mirror axis is the thin black line. Since a mirror symmetry is an involution, the theorem implies that the six line-slopes form a quadrilateral set. Hence again they can be used to form a drawing of a tetrahedron.

Fig. 8.12. Mirroring slopes.
Theorem 8.2 showed that there is a close relation of quadrilateral sets and projective involutions. Any three pairs of images and pre-images of a projective involution on a line form a quadrilateral set. The converse is also true. For this we first observe that if a projective map in \( \mathbb{RP}^1 \) interchanges two points then it will automatically be an involution.

**Lemma 8.1.** Let \( \tau: \mathbb{RP}^1 \to \mathbb{RP}^1 \) be the a projective transformation with \( \tau(a) = a' \) and \( \tau(a') = a \) for distinct points \( a \) and \( a' \) then \( \tau \) is an involution.

**Proof.** W.l.o.g. we may (after a suitable projective transformation) assume that \( a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( a' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Thus the matrix \( T \) that represents \( \tau \) must have the form \( T = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \) for non-zero parameters \( \alpha \) and \( \beta \). Calculating \( T^2 \) we get

\[
T^2 = \begin{pmatrix} \alpha \beta & 0 \\ 0 & \alpha \beta \end{pmatrix} = \alpha \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus \( T^2 \) represents the identity map on \( \mathbb{RP}^1 \). \( \square \)

With this lemma it is easy to prove that every quadrilateral set induces an involution.

**Theorem 8.3.** Let \( a, b, c \) be three distinct points in \( \mathbb{RP}^1 \). Let \( a'b'c' \) be three additional points. If \( (a, a'); (b, b'); (c, c') \) forms a quadrilateral set then the projective map \( \tau \) uniquely defined by \( \tau(a) = a', \tau(b) = b', \tau(c) = c' \) is an involution.

**Proof.** If \( a = a', b = b', c = c' \) then \( \tau \) must be the identity which is trivially an involution. Thus at least one of the pairs consists of different points. Assume w.l.o.g that \( a \) and \( a' \) are distinct. Consider the uniquely defined transformation \( \tau \) that satisfies \( \tau(a) = a', \tau(a') = a, \tau(b) = b' \). By Lemma 8.1 this transformation will be an involution. Thus it suffices to show that \( \tau(c) = c' \). Theorem 8.2 implies that

\[(a, \tau(a)), (b, \tau(b)), (c, \tau(c))\]

form a quadrilateral set. Using our knowledge about \( \tau \) we see that

\[(a, a'), (b, b'), (c, c')\]

is a quadrilateral set. Since five points of a quadrilateral set determine the sixth one uniquely we must have that \( \tau(c) = c' \). This proves the theorem. \( \square \)

So to every quadrilateral set we can associate in a natural way an involution. It is a remarkable fact that the two fixpoints of the involution are in harmonic position two all point pairs in the quadrilateral set.
If $T$ represents a projective transformation then the Eigenvectors of $T$ correspond to the fixpoints of the transformation. For every Eigenvector $p$ of $T$ we have $Tp = \lambda p$ and $p$ is mapped to itself. If $T$ is a $2 \times 2$ matrix, then it may have either two real or two complex conjugate eigenvectors (up to scalar multiples). If the eigenvectors are real then they correspond to points in $\mathbb{RP}^1$ that are invariant under $T$.

**Theorem 8.4.** Let $\tau$ be the involution associated to the points of a quadrilateral set $(a, a'; b, b', c, c')$. Assume that $\tau$ has two distinct real fixpoints $p$ and $q$. Then the point pairs $(p, q)$ and $(a, a')$ form a harmonic set.

**Proof.** If $\tau$ is an involution then $(a, \tau(a); p, \tau(p); q, \tau(q))$ is a quadrilateral set. Using our knowledge on the definition of $\tau$ and the fixpoint properties of $p$ and $q$ we see that $(a, a'; p, p; q, q)$ is a quadrilateral set. Thus we have $[a, p][p, q][q, a'] = [a, q][p, a'][q, p]$. Using the distinctness of $p$ and $q$ we can cancel $[p, q]$ from this expression and are left with $[a, p][q, a'] = [a, q][p, a']$ which is exactly the characterization for a harmonic pair of point pairs. \hfill \Box

Clearly, in the same way the pair of points $(p, q)$ is also harmonic with respect to $(b, b')$ and to $(c, c')$. This leads us to a nice characterization of fixpoints of $\tau$. If $\tau$ is an involution associated to $(a, a'; b, b'; c, c')$ then the fixpoints of $\tau$ are exactly those two points that are simultaneously harmonic to all three point pairs of the quadrilateral set. In fact, already any two of the point pairs of the quadrilateral set determine the position of $p$ and $q$ uniquely.

The fact that $(p, q)$ are also harmonic with respect to the last pair is exactly the quadrilateral set condition.

There is a little subtlety concerning the existence of the fixpoints that will become relevant later in this book: The fixpoints need not to be real. We saw that the fixpoints correspond to the eigenvectors of the matrix $T$ that represents $\tau$. If this matrix has complex eigenvalues (like the involution \(
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\)) then the Eigenvalues are also complex (and cannot made real by multiplication with a scalar). In such a case the transformation has no real fixpoints. This case geometrically is related to “rotation-like” transformations $\tau$ like the $90^\circ$ rotation we used in Figure 8.11. The case in which real fixpoints exists is related to “reflection-like” transformation, like the mirror image operation we used in Figure 8.12.