

— *Classwork* —**Question 1. A concrete Cayley-Klein geometry**

- a) Given the following dual conic B , determine the corresponding primal conic A .

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

- b) Investigate whether the conic described by the primal-dual pair (A, B) is degenerate or not.
If it is degenerate, give coordinates for the components into which it decomposes, both primal and dual.
- c) Now this conic (A, B) shall be used as the fundamental conic of a Cayley-Klein geometry. Describe the resulting measurements for distances and angles respectively as “hyperbolic”, “parabolic” or “elliptic”.
- d) Now assume $c_{\text{ang}} = \frac{1}{2}$. Characterize under which conditions the angle between two lines will be real.
- e) The cross ratio which can be constructed most easily is the harmonic set. use this to construct two lines which enclose a well-defined and real angle. Base your construction on a coordinate system into which the conic (A, B) has been drawn. Also give an explicit description of the value of the angle you are constructing.
- f) Discuss what choices for c_{dist} would be most suitable for distance measurements.
- g) Can you construct two points which have a distance of 2 from one another?
If so, perform that construction.
- h) Given two points, can you construct their midpoint, i.e. a point which has equal distance from either point according to the measurement of this geometry?
If so, choose two points in a sufficiently generic position and perform the construction.

Question 2. Circle mappings

Given the following points in \mathbb{R}^2 :

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad A' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad B' = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad C' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

All of these points are obviously located on the unit circle. Now we are looking for different mappings which leave the unit circle invariant and which map A to A' , B to B' and C to C' . At first you only need to describe how to compute this, the actual computation will then be a homework task.

- a) Interpret the real coordinates as real and imaginary part of a complex number, which you can then treat as a point in \mathbb{CP}^1 . Specify the steps necessary to compute a *projective transformation* which matches the requirements stated above.
- b) Apart from the projective transformation just discussed, there is another *harmonic transformation* in \mathbb{CP}^1 which is not a projective transformation but still satisfies all the stated requirements. Find a description for that as well.
- c) Now embed the stated points into \mathbb{RP}^2 . Here, too, there is a projective transformation which satisfies all stated requirements. Consider how you could obtain that.

Question 2. Circle mappings (continued)

This task is a continuation of the preceding classwork. Here you are asked to explicitly compute the transformations in question.

- d) Compute the projective transformation in \mathbb{CP}^1 .
- e) Compute the other harmonic map in \mathbb{CP}^1 .
- f) Compute the projective transformation in \mathbb{RP}^2 .
- g) Apply all three transformations to the point

$$P = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$$

State the respective results again as dehomogenizes points in \mathbb{R}^2 .

Try to proceed as far as possible using mental arithmetic. Afterwards you can use a computer to conclude or verify your computations. Clearly state how far you were able to proceed without resorting to electronic help.

Question 3. Dual partner

Given the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

as well as some other matrices

$$B_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- a) Decide which of these matrices B_i will form a valid primal-dual pair together with A . Justify your answer using a suitable computation.
- b) Describe these primal and dual conics graphically, both if they form a valid pair and if they do not.
- c) Describe which Cayley-Klein geometries will result from each of the valid pairs.

Question 4. Very degenerate

- a) Choose a degenerate conic where both primal and dual matrix have rank 1. State the matrices explicitly, and draw the resulting conic in a coordinate system.
- b) With respect to this Cayley-Klein geometry, construct a sequence of at least five equidistant points on an arbitrary line.
- c) Also construct a pencil of at least five equiangular lines through a common point. Two consecutive lines of that sequence should therefore form the same angle with one another.