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Projective Geometry SS 2016

www-m10.ma.tum.de/ProjektiveGeometrieSS16

Solutions for Worksheet 4 (2016-05-05)

— ***Classwork*** —

Question 1. Rational projective plane

a) Argue briefly that $\mathbb{QP}^2 = (\mathcal{P}_\mathbb{Q}, \mathcal{L}_\mathbb{Q}, \mathcal{I}_\mathbb{Q})$ is a projective plane.

b) State the transformation matrix M which expressed a rotation around the origin by an angle of 45° counter-clockwise.

c) The set of rational points shall now be combined with a rotated version of itself, and the result shall be supplemented to form a projective plane. The result is a structure which is a part of $\mathbb{RP}^2$, and the operations $\vee$ and $\wedge$ will be interpreted in that real plane.

$$\mathcal{P}_0 := \mathcal{P}_\mathbb{Q} \cup \{M \cdot p \mid p \in \mathcal{P}_\mathbb{Q}\} \qquad \mathcal{L}_0 := \{a \vee b \mid a, b \in \mathcal{P}_0\}$$
$$\mathcal{P}_{i+1} := \{a \wedge b \mid a, b \in \mathcal{L}_i\} \qquad \mathcal{L}_i := \{a \vee b \mid a, b \in \mathcal{P}_i\}$$
$$\mathcal{P} := \bigcup_{i=0}^{\infty} \mathcal{P}_i \qquad \mathcal{L} := \bigcup_{i=0}^{\infty} \mathcal{L}_i$$

Examine whether a finite number of iterations of the above will already lead to a fixed point. Phrased differently, whether there exists some $i < \infty$ such that $\mathcal{P}_i = \mathcal{P}_{i+1}$ und $\mathcal{L}_i = \mathcal{L}_{i+1}$.

d) Demonstrate briefly that the sets $\mathcal{P} \subseteq \mathcal{P}_\mathbb{R}$ and $\mathcal{L} \subseteq \mathcal{L}_\mathbb{R}$ constructed above will form a projective plane if taken together with the incidence relation of the real projective plane, restricted to these objects:

$$\mathcal{I} := (\mathcal{P} \times \mathcal{L}) \cap \mathcal{I}_\mathbb{R}$$

e) Try to describe this plane as a projective plane over some number field. If you find a suitable field, write it down using common notation.

f) In the lecture it has been shown that in $\mathbb{RP}^2$, every collineation is a projective transformation. Find out where that proof will fail for the plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ constructed above.

g) Prove or disprove the statement that every collineation is a projective transformation in the plane constructed above, Solution:

a) $\mathbb{Q}$ is a field. And we can construct a projective plane over every field.

   As the operations Join and Meet are polynomials in the coordinates of points and lines, the field $\mathbb{Q}$ is never left.

b) A possible representative is

$$M = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

   Multiplication with this matrix brings $\sqrt{2}$ into the first two coordinates. The last coordinate stays rational.

c) Central to thsi part is the definition. The analysis of finiteness is less important. It will be consider here nevertheless.

   Iteration ends after finitely many steps. The proof is easy, when, first, one does not try to find the minimal i at which the set is saturated and, second, one suspects the result form part e).

   Even at the start, the x-axis contains all points of the form $(x, 0, 1)^T$ both for $x \in \mathbb{Q}$ and $x \in \sqrt{2} \cdot \mathbb{Q}$. Using a von-Staudt construction, we can add all those points to get every point with $x = a + \sqrt{2}b$ for $a, b \in \mathbb{Q}$. And we need only finitely many geometric arithmetic operations for that. So, at the latest ehen the iteration depth reaches this number, all necessary elements including the final results will be included. On the y-axis we get all elements of $\mathbb{Q}\left[\sqrt{2}\right]$ as y-coordinates by the same argument. From points on the axes we get all points in the plane, again in fintely many steps. This also results in all directions of lines and all intersections with the line at infinity.

d) The esxistence of Join and Meet follows from the construction: whenever we can combine 2 lines or points in one of the sets, the respective combination will get included into the set in the next iteration step. The uniqueness of Join and Meet and the existence of 4 points in general position get inherited from $\mathbb{R}P^2$.

e) We get the projective plane $\mathbb{Q}\big[\sqrt{2}\big]\mathbb{P}^2$, with $\mathbb{Q}\big[\sqrt{2}\big] = \big\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\big\}$.

f) The proof in the lecture definied "positive" using squares: A number x is positive, iff there exists a number y with $y^2 = x$. This characterisation fails in $Q\big[\sqrt{2}\big]$, as e.g. $\sqrt{2}$ is no square of elements of the field.

g) The proposition is wrong as there is a non-trivial field automorphism:

$$\varphi : a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

   That this is indeed a field homomorphism is easily varified:

$$\begin{aligned} \varphi(a + b\sqrt{2}) + \varphi(c + d\sqrt{2}) &= (a - b\sqrt{2}) + (c - d\sqrt{2}) = (a + c) - (b + d)\sqrt{2} &= \varphi\big((a + c) + (b + d)\sqrt{2}\big) \\ \varphi(a + b\sqrt{2}) \cdot \varphi(c + d\sqrt{2}) &= (a - b\sqrt{2}) \cdot (c - d\sqrt{2}) = (ac + 2bd) - (ad + bc)\sqrt{2} &= \varphi\big((ac + 2bd) + (ad + bc)\sqrt{2}\big) \end{aligned}$$

   Applied to all coordinates, a field automorphism preserves collinearities, but it cannot be written as a projective transformation. The first part holds, as incidences are defined via the scalar product and its vanishing is preserved by automorphisms. The second part holds, since the standard basis (with only rational coordinates) gets mapped to itself with the map not being the identity.

Question 2. The whole plane

We want to consider the plane $\mathbb{Z}\mathbb{P}^2$. The usual definition using scalra multiples will not work over arbitrary rings (or rather integral domains). So we do the following:

For elements $P, Q \in \mathbb{Z}^3 \setminus \{0\}$ we define

$$P \sim Q \quad :\Leftrightarrow \quad \exists \lambda, \mu \in \mathbb{Z}\backslash\{0\} : \ \lambda P = \mu Q.$$

Then we set

$$\mathcal{P}_{\mathbb{Z}} := {}^{\mathbb{Z}^3\backslash\{0\}}/_{\sim}.$$

I.e. $\mathbb{Z}^3 \setminus \{0\}$ modulo the relation $\sim$.

We define $\mathcal{L}_{\mathbb{Z}}$ analogously and the the incindence relation $\mathcal{I}_{\mathbb{Z}}$ via the scalar product, as usual.

a) Is the structure $(\mathcal{P}_{\mathbb{Z}}, \mathcal{L}_{\mathbb{Z}}, \mathcal{I}_{\mathbb{Z}})$ defined above in fact a projective plane? Is the relation $\sim$ an equivalence relation at all?

b) Is there a projective plane over some field which is isomorphic to the structure above? If so, name the underlying field. If not, prove that no such field can exist.

c) Find an incidence configuration which cannot be realized as a part of the above structure, but can be realized in $\mathbb{RP}^2$.

<u>SOLUTION:</u>

a) The given relation is an equivalence relation. If it was defined with only one scalar factor (as in the case of fields), this would not be true. For example, the symmetry would not hold, since not all integers are invertible. The plane is indeed a projective plane. One can define Join and Meet using cross product, as their computation do not need any divisions. And the standard basis consisting of 4 points in general position has only integer coordinates.

b) The given plane is isomorphic to $\mathbb{QP}^2$. Homogeneous coordinate vectors in $\mathbb{Q}^3$ can be scaled with the lowest common denominator to obtain a vector from $\mathbb{Z}^3$ in the same equivalence class.

c) One can use the construction from exercise 3c) validating $\sqrt{2}$. Using the cross-ratios

$$(0, \infty; \sqrt{2}, -\sqrt{-2}) = -1 = (-\sqrt{2}, \sqrt{2}; 1, 2),$$

we deduce that the respective constructions of harmonic conjugated points is not realisable. Even when no projective basis is given but arbitrarily chosen, there exists a quadrupel of points with cross-ratio $\sqrt{2}$. As cross-ratios in $\mathbb{Q}P^2$ have to be rational, this configuration does not exist here.

— ***Homework***—

Question 3. Geometrically computable operations

For three points A, B, C on a projective line, the function h shall be defined by

$$h(A, B; C) = D \quad \Leftrightarrow \quad (A, B; C, D) = -1$$

So this function h computes a fourth point in such a way that it forms a harmonic set with the given three points. Furthermore, the points 0, 1 and ∞ shall be designated on the line of computation, as well as two more points x and y.

a) Find out which of the following points on the line of computation can be constructed from the given points by repeated application of the function h. For those where such a construction is possible, state a formula to obtain that point. For those where no such construction is possible, give a reason for this fact.

(1) $3 \cdot x$
(2) $x + y$
(3) $x - y$
(4) $x \cdot y$
(5) $\frac{1}{x}$
(6) $\frac{x}{y}$
(7) $\sqrt{x}$
(8) x^2
(9) e^x

Note: Operations which you have already reduced to applications of h can be used in subsequent definitions as a kind of shorthand notation. So you don't really have to state every single expression as a nested expression of h applications, as long as you have ensured that such a notation would be possible in theory.

b) The function h is not defined if two of the three points A, B, C coincide. Investigate which of the expressions are affected by this case, and give a case distinction which could handle the required computation in these special cases. You may assume that 0, 1 and ∞ are pairwise distinct.

c) Suppose someone were to denote yet another point, claiming it had position $\sqrt{2}$ with respect to the projective scale in question. Can you verify this claim using some incidence configuration? If not, why is this impossible? If you can verify the point, can you also use that construction to construct $\sqrt{2}$.

Solution:

a) Except for (7) and (9), all operations are constructible from nested harmonic conjugation.

$$
\begin{aligned}
2 \cdot x &= h(x, \infty; 0) \\
3 \cdot x &= h(2 \cdot x, \infty; x) = h(h(x, \infty; 0), \infty; x) && (1) \\
\frac{x+y}{2} &= h(x, y; \infty) \\
x + y = 2 \cdot \frac{x+y}{2} &= h(h(x, y; \infty), \infty; 0) && (2) \\
-y &= h(0, \infty; y) \\
x - y = x + (-y) &= h(h(x, h(0, \infty; y); \infty), \infty; 0) && (3) \\
x^2 &= h(-x, x; 1) = h(h(0, \infty; x), x; 1) && (8) \\
\frac{x}{2} &= h(0, x; \infty) \\
x \cdot y &= \frac{1}{2}\left((x+y)^2 - (x^2+y^2)\right) = \frac{(x+y)^2 + (-(x^2+y^2))}{2} && (4) \\
&= h\Big(\underbrace{h\big(\underbrace{h(0,\infty;\underbrace{h(h(x,y;\infty),\infty;0)}_{x+y})}_{-(x+y)},\underbrace{h(h(x,y;\infty),\infty;0)}_{x+y};1\big)}_{(x+y)^2},\underbrace{h(0,\infty;\underbrace{h(h(\underbrace{h(h(0,\infty;x),x;1)}_{x^2},\underbrace{h(h(0,\infty;y),y;1)}_{y^2};\infty),\infty;0)}_{x^2+y^2})}_{-(x^2+y^2)};\infty\Big) \\
\frac{1}{x} &= h(-1, 1; x) = h(h(0, \infty; 1), 1; x) && (5) \\
\frac{x}{y} &= x \cdot \frac{1}{y} && (6)
\end{aligned}
$$

Using the operations (7) and (9), it would be possible to construct an irrational result from purely rational input—for example from $x = 2$. As harmonic conjugates can be constructed over $\mathbb{Q}P^2$, too, such a construction must yield a rational result if it started with rational points. Therefore, these two operations cannot be recreated by incidence constructions.

b) The following cases have to be handled seperately for finite x and y:

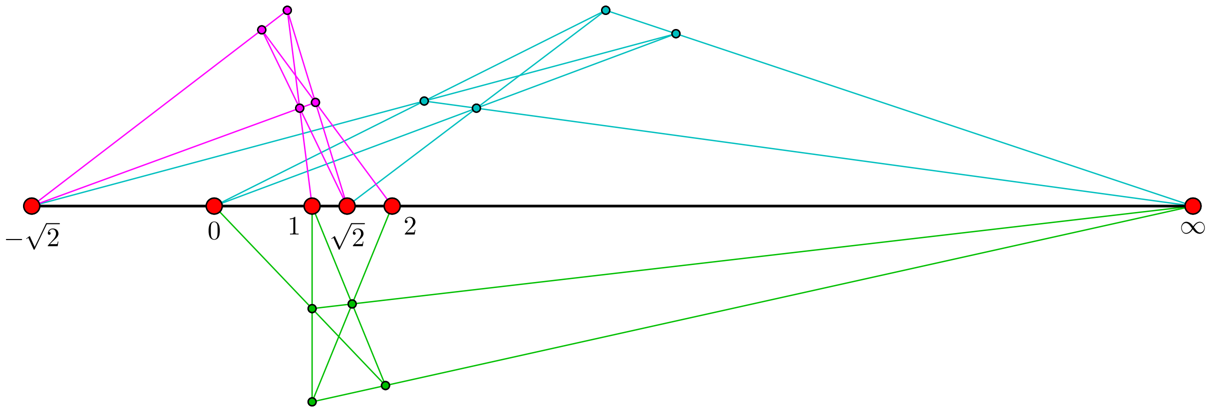
- When $x = 0$, we have $2 \cdot x = 0$ and $3 \cdot x = 0$.
- When $x = y$, then $\frac{x+y}{2} = x = y$.
- When $\frac{x+y}{2} = 0$ and thus $x = -y$, we have $x + y = 0$.
- When $y = 0$, then $-y = 0$ and $x - y = x$.
- When $|x| = 1$, so $x^2 = 1$.
- When $x = 0$, then $x^2 = 0$ and $\frac{x}{2} = 0$.
- When $x = 1$, then $\frac{x}{1} = 1$.
- For $x \cdot y$ and $\frac{x}{y}$ we have to use the above rules in every single step.

c) We cannot construct $\sqrt{2}$, as otherwise we would be able to construct an irrational point from rational points, contradicting part a).

But when such a point is given, we can verify whether its square really equals 2:

$$
\begin{aligned}
-\sqrt{2} &= h(0, \infty; \sqrt{2}) \\
\left(\sqrt{2}\right)^2 &= h(-\sqrt{2}, \sqrt{2}; 1) \\
2 \cdot 1 &= h(1, \infty; 0)
\end{aligned}
$$

When the point $\left(\sqrt{2}\right)^2$ coincides with the point $2 \cdot 1$, then $\sqrt{2}$ really is a number with square 2. Over the real numbers, we have possible points for that: $\pm\sqrt{2}$. So, up to its sign, we can recognise the square-root. Considering the square-root as multi-valued function, the property of being $\sqrt{2}$ can be perfectly checked.



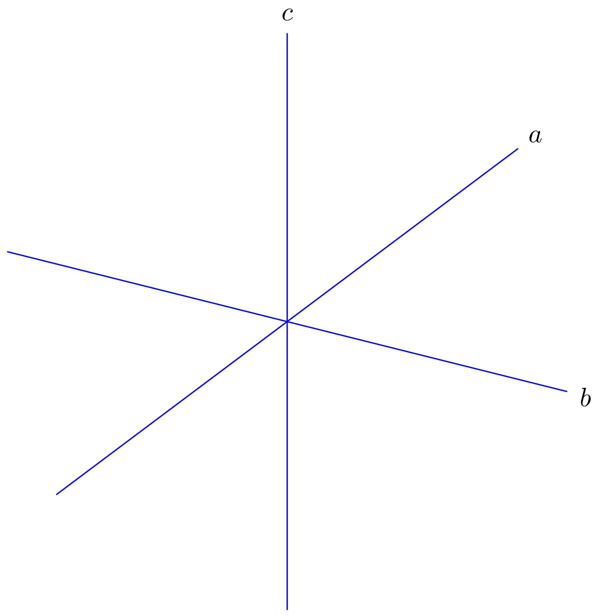
Question 4. Computations using slopes

If you only consider the slopes of the lines in the projective plane, these will form a projective line: all real numbers can occur as slopes, and in addition to these there are lines with infinite slope.

a) Create a draft explaining how to construct the point D which forms a harmonic set with three given points A, B, C, i.e. the point D such that $(A, B; C, D) = -1$. Write a step-by-step description of this construction.

b) Dualize this construction description.

c) Use this dual construction to construct the harmonic line d for the following three lines $a, b; c$.



d) Show that the cross ratio of the four lines you just constructed can also be written as a fraction of determinants of homogeneous coordinates of the lines involved.

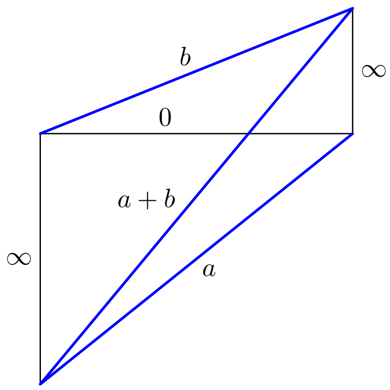
e) Consider in general the cross ratio of four arbitrary lines through a common point, computed as we just did using the homogeneous coordinates of the lines. Show that the points of intersection of these lines with the line at infinity have the same cross ratio.

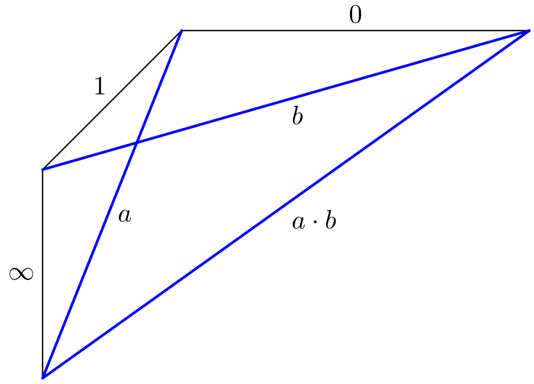
f) Intersect the four concurrent lines in your construction for sub-task c) with another (finite) line. Check, using a suitable construction, whether the four points of intersection form a harmonic set.

g) Explain how the cross ratio of four slopes could be defined in a sensible way even if the four lines in question don't pass through a common point.

h) Lines with slopes 0, 1 and ∞ fix a projective scale. Check that with respect to this scale, measuring slopes via the coordinate vector of a line leads to the same result as computing the slope via the cross ratio relative to the basis stated above.

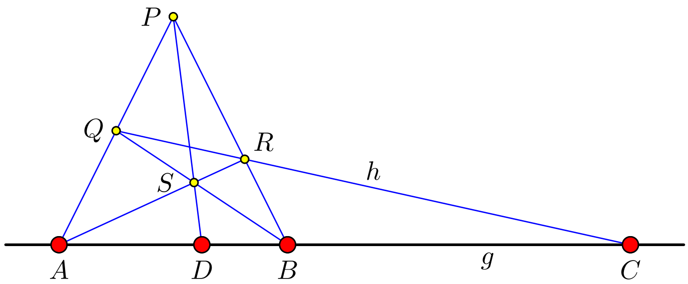
i) Prove that the following constructions can actually be used to add and multiply slopes of lines, the way the labels suggest. You may use affine or Euclidean arguments.





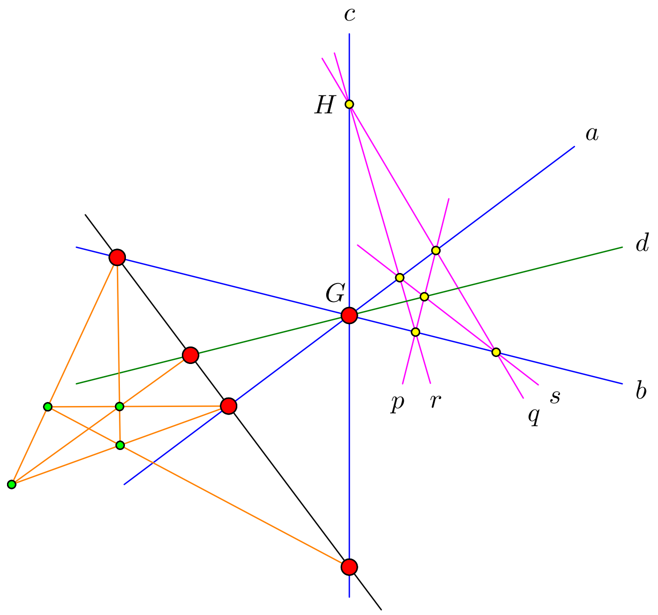
<u>Solution:</u>

a)
1. Choose point P arbitrarily, not on the line g.
2. Choose line h through C arbitrarily.
3. $Q = (A \vee P) \wedge h$.
4. $R = (B \vee P) \wedge h$.
5. $S = (A \vee R) \wedge (B \vee Q)$.
6. $D = (P \vee S) \wedge g$.



b)
1. Choose line p arbirtrarily, not going through G.
2. Choose point H on c arbtrarily.
3. $q = (a \wedge p) \vee H$.
4. $r = (b \wedge p) \vee H$.
5. $s = (a \wedge r) \vee (b \wedge q)$.
6. $d = (p \wedge s) \vee G$.

c) The primal configuration consists of 4 auxilary points and their 6 connections, but the dual has 4 auxiliary lines and their 6 intersections.



On the right you see the constrution for part c); on the left you already see the construction for part f).

d) The algebraic computations for harmonic conjugation are not influenced by objects being primal or dual: Starting from certain vectors, new vectors get computed using the cross product. Whence, the interpretation of the configuration as primal or dual is independent from the underlying algebraic structure. When the primal configuration leads to a certain equation for determinants, the dual one has to result in the same equation. So, in that case we have

$$(a, b; c, d)_l = \frac{[a, c, l][b, d, l]}{[a, d, l][b, c, l]} = -1$$

for arbitrary lines l not incident to the point G, too.

e) For 4 different intersection points on the line at infinity to exist, the common point G cannot lie at infinity. It hence is appropriate to look at the 4 lines from the line at infinity in order to compute the cross-ratio with the term above:

$$(a, b; c, d)_{l_\infty} = \frac{[a, c, l_\infty][b, d, l_\infty]}{[a, d, l_\infty][b, c, l_\infty]}$$

A line g with homogeneous coordinates $(g_1, g_2, g_3)^T$ intersects the line at infinity $l_\infty = (0, 0, 1)^T$ in a point

$$g \times l_\infty = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g_2 \\ -g_1 \\ 0 \end{pmatrix}$$

All those intersection points can be looked at from an arbitrary finite point like $(0, 0, 1)^T$. For each matrix in the cross-ratio we then have an equation of the form

$$\begin{vmatrix} a_1 & c_1 & 0 \\ a_2 & c_2 & 0 \\ a_3 & c_3 & 1 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 & 0 \\ -a_1 & -c_1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

This equation holds as the permuation of the rows is compensated by a sign change and as the last row does not affect the first two columns. When the value of each determinat does not change—no matter whether one looks at the line from the line at infinity or at the respective point from the origin—then this is also true for every term consisting of these determinants and hence for the cross-ratio.

Therefore, we can define the cross-ratio of 4 lines via the intersection with another line or via determinants consisting of the lines' coordinates. The results will be the same.

f) The intersections form an harmonic set, which can easily be shown using the usual construction. The result, by the way, gives a nice incidence theorem, saying that the given construction must work. You can see the construction in part c).

g) The representative of all line with a certain direction is the point at infinity in that direction. Focusing solely on slopes means working with this point and furthermore working on a projecitve line—the line at infinity.

So, while being able to compute a cross-ratio for 4 concurrent lines by looking at them from an arbirtrary (non-concurrent) line, one chooses the line at infinity as this additional line when focusing on the slopes.

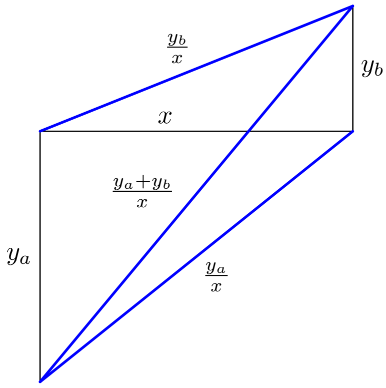
h) A line with homogeneous coordinates $(a, b, c)^T$ can be decribed by the non-homogeneous equation $ax+by+c=0$ or $y = -\frac{a}{b} \cdot x - \frac{c}{b}$, with $-\frac{a}{b}$ being its slope. One could use the following line to represent the projective scale:

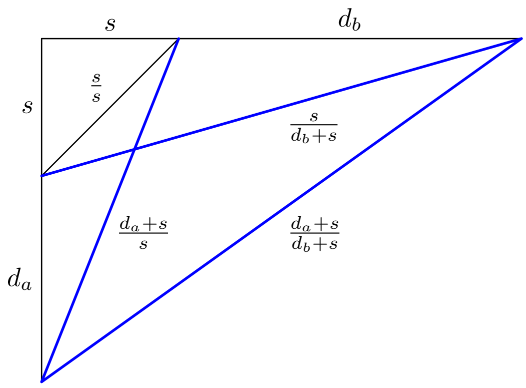
$$g_0 = \begin{pmatrix} 0 \\ 1 \\ * \end{pmatrix} \qquad g_1 = \begin{pmatrix} 1 \\ -1 \\ * \end{pmatrix} \qquad g_\infty = \begin{pmatrix} 1 \\ 0 \\ * \end{pmatrix}$$

W.r.t. this scale, one can now plug a line with slope t into the cross-ratio:

$$(g_0, g_\infty; g_t, g_1)_{l\infty} = \frac{[g_0, g_t, l_\infty][g_\infty, g_1, l_\infty]}{[g_0, g_1, l_\infty][g_\infty, g_t, l_\infty]} = \frac{\begin{vmatrix} 0 & t & 0 \\ 1 & -1 & 0 \\ * & * & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ * & * & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ * & * & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & t & 0 \\ 0 & -1 & 0 \\ * & * & 1 \end{vmatrix}} = \frac{(-t)\cdot(-1)}{(-1)\cdot(-1)} = t$$

i) One can simply label the lengths of the horizontal and vertical lines and give the slopes of the other lines as fractions in these variables:





This immediately yields:

$$a + b = \frac{y_a}{x} + \frac{y_b}{x} = \frac{y_a + y_b}{x}$$
$$a \cdot b = \frac{d_a + s}{s} \cdot \frac{s}{d_b + s} = \frac{d_a + s}{d_b + s}$$