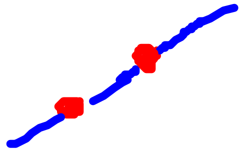
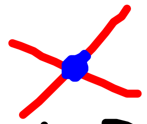
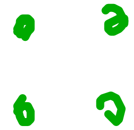
So far:

- Axioms: A1, A2, A3
- First conclusions: e.g. $n^2+n+1=|\mathcal{P}|$
- Class of models for A1 A2, A3

Projective Planes over a field

$\mathbb{K}$ arbitrary field

$$\mathcal{P} = \frac{\mathbb{K}^3-\{0\}}{\mathbb{K}-\{0\}} = \{[p] \mid p\in\mathbb{K}^3-\{0\}\}$$

$$[p] = \{\lambda p \mid \lambda \in \mathbb{K}-\{0\}\}$$

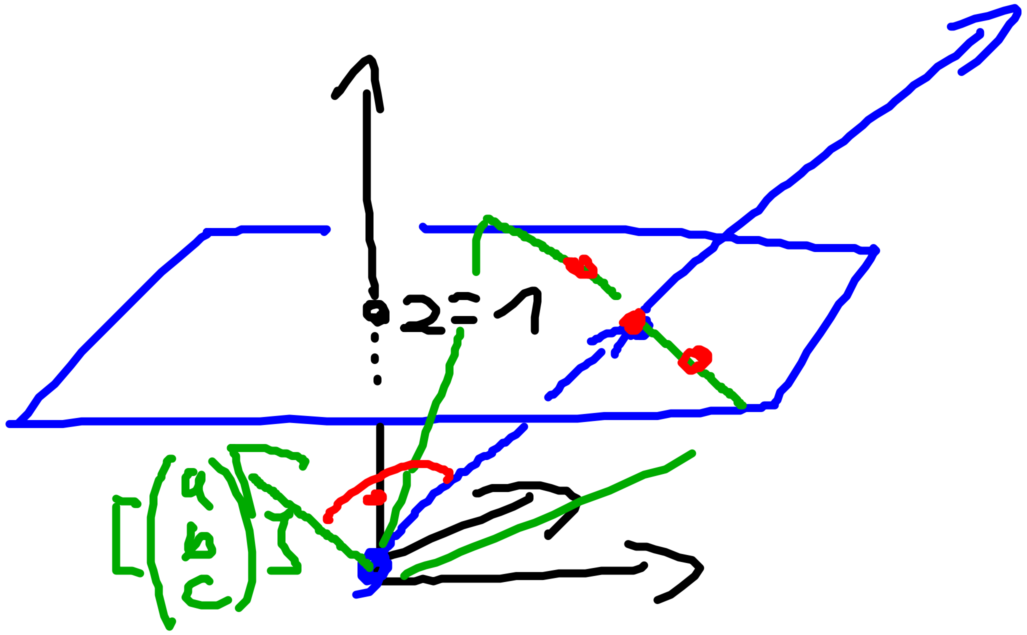
$$\mathcal{L} = \frac{\mathbb{K}^3-\{0\}}{\mathbb{K}-\{0\}}$$

$$[p] \; I \; [\ell] \iff \langle p,\ell\rangle = 0$$

$$\boxed{\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b\in\mathbb{Q}\}}$$

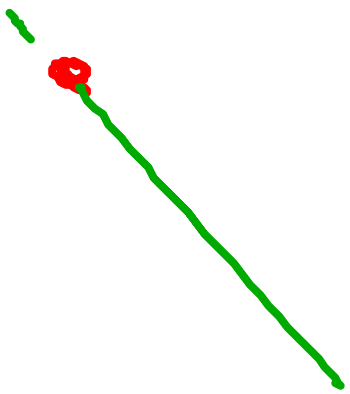
$$\mathbb{R}^2 \to P_{\mathbb{R}}$$

$$\begin{pmatrix}x\\y\end{pmatrix} \to \left[\begin{pmatrix}x\\y\\1\end{pmatrix}\right]$$

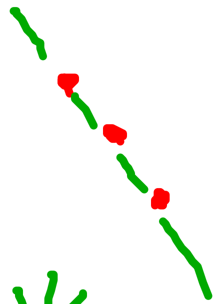



For any Field

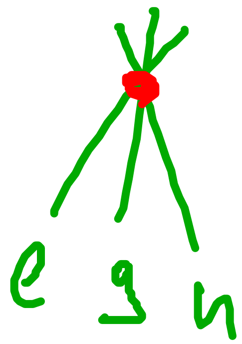
- Incidence: $[p]\ I_{\mathbb{R}}\ [\ell] \Leftrightarrow \langle p, \ell \rangle = 0$



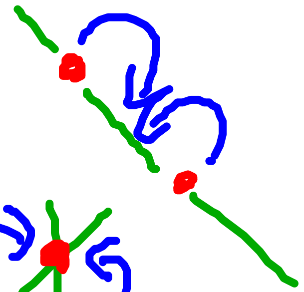
- collinearity: $[p],[q],[r]$ collinear $\Leftrightarrow$ $\det(p, r, q) = 0$



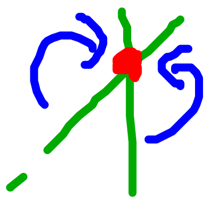
- concurrence of lines: $[\ell],[g],[h]$ concurrent $\Leftrightarrow$ $\det(\ell, g, h) = 0$




- join: The join of $[p],[q]$ is $[p \times q]$



- Meet: Meet of $[\ell][m]$ is $[\ell \times m]$



**Def** Collineation of proj Plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a map $\tau : (\mathcal{P} \cup \mathcal{L}) \to \mathcal{P} \cup \mathcal{L}$, $\tau(\mathcal{P}) = \mathcal{P}$, $\tau(\mathcal{L}) = \mathcal{L}$ bijective and $p \,\mathcal{I}\, \ell \Leftrightarrow \tau(p) \,\mathcal{I}\, \tau(\ell)$

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**Thm:** every collineation is already determined by the images of the points

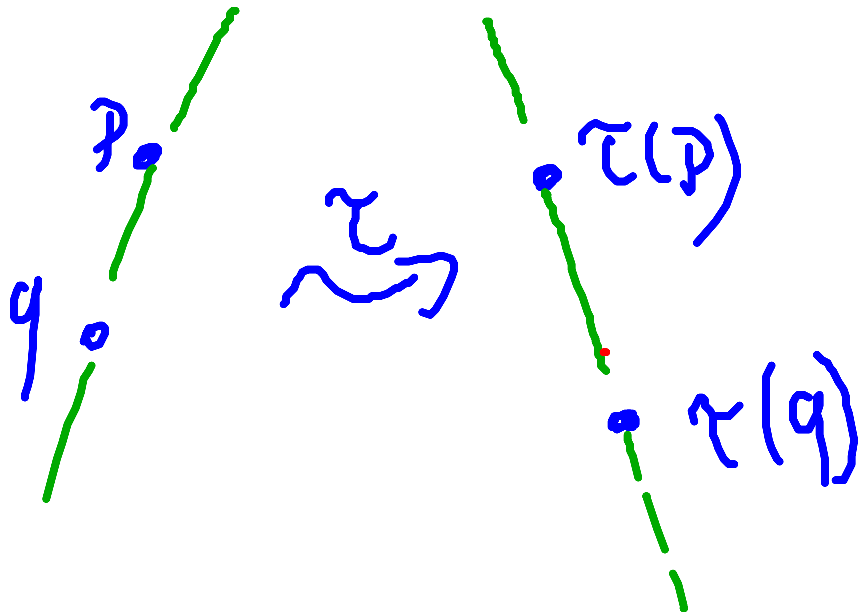
**Proof:** Let $\tau$ be a collineation, $\ell \in \mathcal{L}$

wlog $\ell = p \vee q$, $p \in \mathcal{P}$, $q \in \mathcal{P}$, $p \neq q$

$\Rightarrow p \,\mathcal{I}\, \ell$, $q \,\mathcal{I}\, \ell$

$\Rightarrow \tau(p) \,\mathcal{I}\, \tau(\ell)$, $\tau(q) \,\mathcal{I}\, \tau(\ell)$

$\Rightarrow \tau(p) \vee \tau(q) = \tau(\ell)$



**Def:** Projective Transformation for $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$

Let $A \in \mathbb{K}^{3\times 3}$ be invertable

$$\tau_A\big|_{\mathcal{P}_{\mathbb{K}}} : \mathcal{P}_{\mathbb{K}} \to \mathcal{P}_{\mathbb{K}}, \quad [p] \mapsto [A\cdot p] \qquad \tau_A\big|_{\mathcal{L}_{\mathbb{K}}} : \mathcal{L}_{\mathbb{K}} \to \mathcal{L}_{\mathbb{K}}, \quad [\ell] \mapsto [(A^{-1})^T\cdot \ell]$$

**Thm:** every projective Transformation is a collineation.

Proof: $\tau_A$ is bijective since $A$ is invertible

$\tau_A$ also preserves incidence:

$$[p]\,\mathcal{I}\,[\ell] \Leftrightarrow \langle p,\ell\rangle = 0 \Leftrightarrow p^T\cdot \ell = 0$$

$$\Leftrightarrow p^T\cdot E\cdot \ell = 0$$

$$\Leftrightarrow p^T\cdot \overbrace{A^T\cdot (A^T)^{-1}}^{E}\cdot \ell = 0 \Leftrightarrow (A\cdot p)^T\cdot (A^{-1})^T \ell = 0$$

$$\Leftrightarrow [Ap]\,\mathcal{I}\,[(A^{-1})^T \ell]$$

For $(\mathcal{P}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \underset{\sim}{I}_{\mathbb{R}})$ (the real proj plane)
every collineation is a projective Transformation!

In general the following holds:

Every collineation of $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, I_{\mathbb{K}})$ is the composition of a projective Transformation and a field automorphism.

$\varphi: \mathbb{K} \to \mathbb{K}$ bijective

$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

$\varphi(a+b) = \varphi(a) + \varphi(b)$

Example:

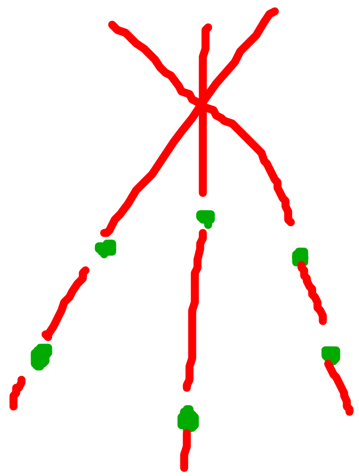
$\varphi: \mathbb{C} \to \mathbb{C}$

$z \to \bar{z}$

$\overline{a \cdot b} = \bar{a} \cdot \bar{b}$

$\overline{a+b} = \bar{a} + \bar{b}$

Def: n-Points are in „general position" if no three them are collinear.



Thm: in $(\mathbb{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \hat{I}_{\mathbb{K}})$ the following holds:

let $a, b, c, d$ be in general pos

and $a', b', c', d'$ " " "

Then there exists a unique projective Transformation with $\tau(a) = a'$, $\tau(b) = b'$, $\tau(c) = c'$, $\tau(d) = d'$

Proof: Geometrie kal Küche

You can assume $\underbrace{\text{w. l. o. g.}}_{\text{up to proj Trafo}}$

- one special point is an infinite Point
- two special points are infinite Points
- a special line is the line at Infinity
- four special point in general Position have special coordinates

„Einschub": Incidence theorems and proof-strategies.

Incidence Theorem is a thm. that uses <u>only</u> statements about points, lines, incidences (no circles, no conics, no lengths, no angels)

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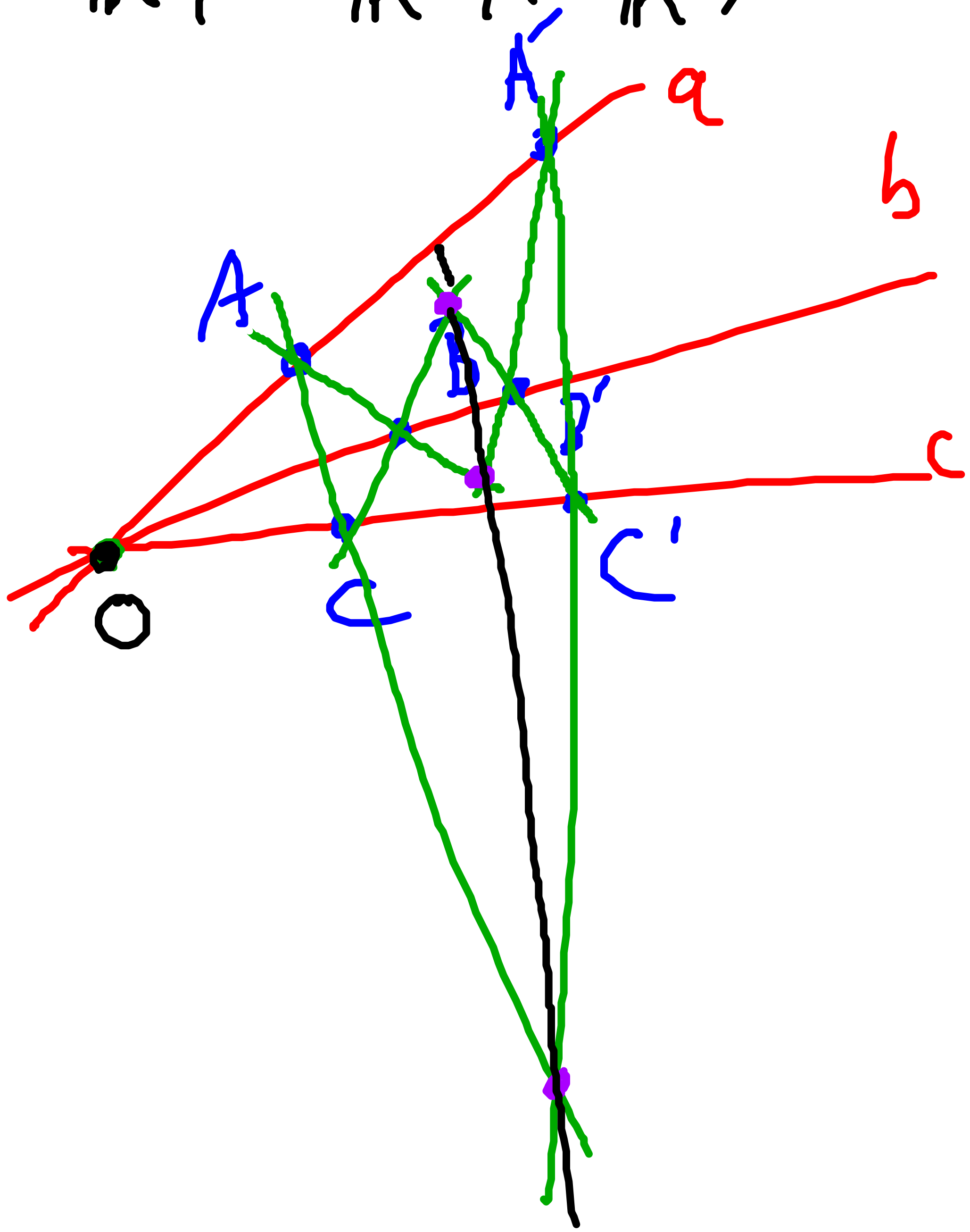
Structure of such Thms:

Hypotheses: Incidences: certain pts are on certain lines

+ Nondegeneracy assumptions certain incidences are <u>not</u> satisfied

Conclusion: some spezial incidence

# Desargues Theorem in $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \tilde{I}_{\mathbb{K}})$




Let $a, b, c$ three different lines through a point $O$

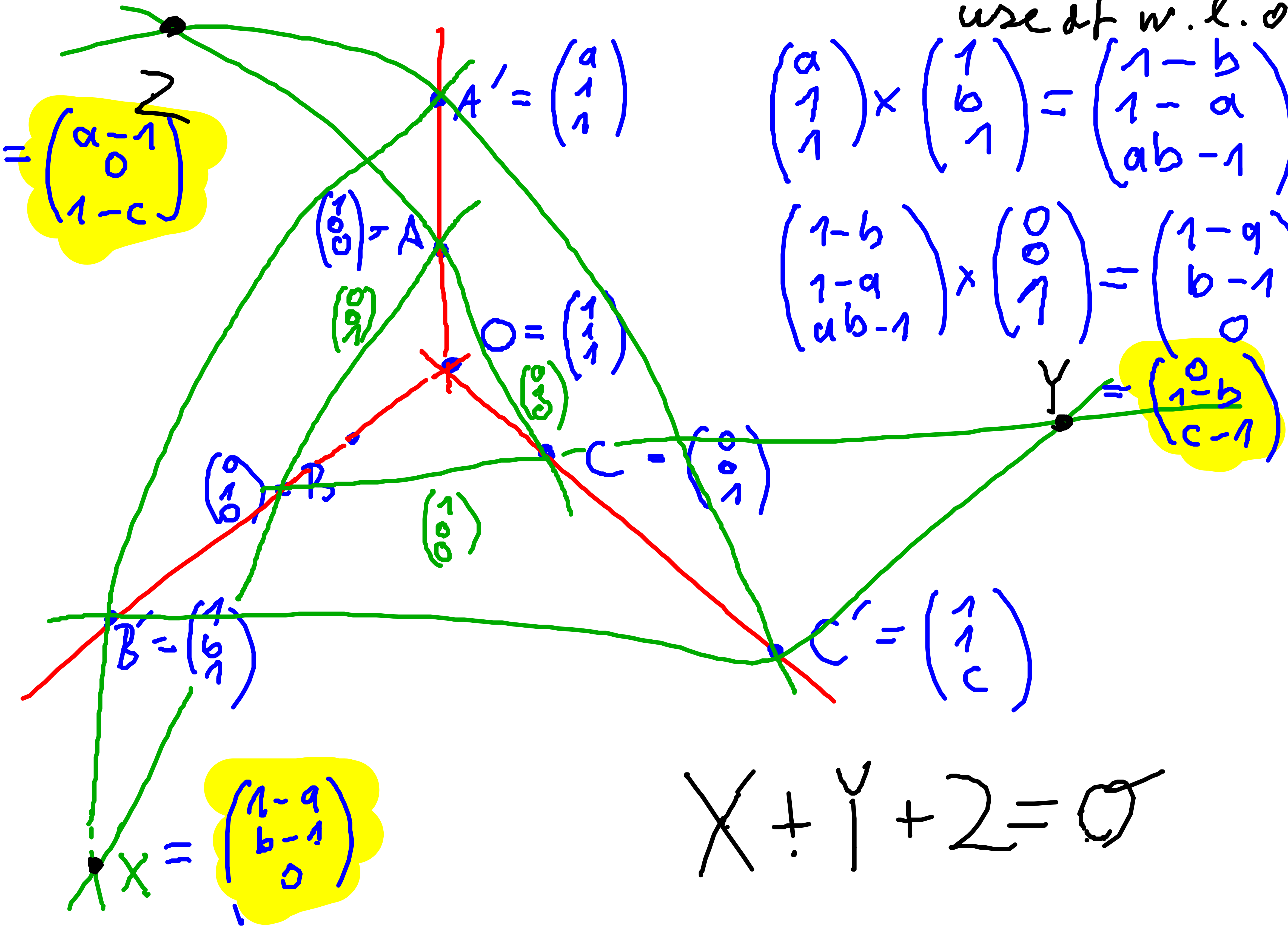
Let $A, A'$ be on $a$
$B, B'$ be on $b$
$C, C'$ be on $c$

and $(A, A', B, B', C, C', O)$ all different

Then:
$(A \vee B) \wedge (A' \vee B')$
$(A \vee C) \wedge (A' \vee C')$
$(B \vee C) \wedge (B' \vee C')$

are collinear!

# 1. Proof strategy: Brute force (with intelligent use of w.l.o.g)




$$\begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} 1-b \\ 1-a \\ ab-1 \end{pmatrix}$$

$$\begin{pmatrix} 1-b \\ 1-a \\ ab-1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-a \\ b-1 \\ 0 \end{pmatrix} = X$$

$$X + Y + Z = 0$$