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Projective Geometry SS 2014

www-m10.ma.tum.de/ProjektiveGeometrieSS14

Worksheet 10 (30. June 2014)

— ***Classwork***—

Question 1. A concrete Cayley-Klein geometry

a) Given the following dual conic B, determine the corresponding primal conic A.

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

b) Investigate whether the conic described by the primal-dual pair (A, B) is degenerate or not.
If it is degenerate, give coordinates for the components into which it decomposes, both primal and dual.

c) Now this conic (A, B) shall be used as the fundamental conic of a Cayley-Klein geometry. Describe the resulting measurements for distances and angles respectively as "hyperbolic", "parabolic" or "elliptic".

d) Now assume $c_{\text{ang}} = \frac{1}{2}$. Characterize under which conditions the angle between two lines will be real.

e) The cross ratio which can be constructed most easily is the harmonic set. use this to construct two lines which enclose a well-defined and real angle. Base your construction on a coordinate system into which the conic (A, B) has been drawn. Also give an explicit description of the value of the angle you are constructing.

f) Discuss what choices for c_{dist} would be most suitable for distance measurements.

g) Can you construct two points which have a distance of 2 from one another?
If so, perform that construction.

h) Given two points, can you construct their midpoint, i.e. a point which has equal distance from either point according to the measurement of this geometry?
If so, choos two points in a sufficiently generic position and perform the construction.

Question 2. Circle mappings

Given the following points in $\mathbb{R}^2$:

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad A' = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad B' = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad C' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

All of these points are obviously located on the unit circle. Now we are looking for different mappings which leave the unit circle invariant and which map A to A', B to B' and C to C'. At first you only need to describe how to compute this, the actual computation will them be a homework task.

a) Interpret the real coordinates as real and imaginary part of a complex number, which you can then treat as a point in $\mathbb{CP}^1$. Specify the steps neccessary to compute a *projective transformation* which matches the requirements stated above.

b) Apart from the projective transformation just discussed, there is another *harmonic transformation* in $\mathbb{CP}^1$ which is not a projective transformation but still satisfies all the stated requirements. Find a description for that as well.

c) Now embed the stated points into $\mathbb{RP}^2$. Here, too, there is a projective transformation which satisfies all stated requirements. Consider how you could obtain that.

Question 2. Circle mappings (continued)

This task is a continuation of the preceeding classwork. Here you are asked to explicitely compute the transformations in question.

d) Compute the projective transformation in $\mathbb{CP}^1$.

e) Compute the other harmonic map in $\mathbb{CP}^1$.

f) Compute the projective transformation in $\mathbb{RP}^2$.

g) Apply all three transformations to the point

$$P = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2$$

State the respective results again as dehomogenizes points in $\mathbb{R}^2$.

Try to proceed as far as possible using mental arithmetic. Afterwards you can use a computer to conclude or verify your computations. Clearly state how far you were able to proceed without resorting to electronic help.

Question 3. Dual partner

Given the following matrix

$$A = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$$

as well as some other matrices

$$B_1 = \begin{pmatrix}0 & 0 & 1\\0 & 0 & 1\\1 & 1 & 2\end{pmatrix} \qquad B_2 = \begin{pmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{pmatrix} \qquad B_3 = \begin{pmatrix}0 & -1 & 1\\-1 & 0 & 1\\1 & 1 & -2\end{pmatrix} \qquad B_4 = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

a) Decide which of these matrices B_i will form a valid primal-dual pair together with A. Justify your answer using a suitable computation.

b) Describe these primal and dual conics graphically, both if they form a valid pair and if they do not.

c) Describe which Cayley-Klein geometries will result from each of the valid pairs.

Question 4. Very degenerate

a) Choose a degenerate conic where both primal and dial matrix have rank 1. State the matrices explicitely, and draw the resulting conic in a coordinate system.

b) With respect to this Cayley-Klein geometry, construct a sequence of at least five equidistant points on an arbitrary line.

c) Also construct a pencil of at least five equiangular lines through a common point. Two consecutive lines of that sequence should therefore form the same angle with one another.