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Projective Geometry SS 2014

www-m10.ma.tum.de/ProjektiveGeometrieSS14

Worksheet 7 (2. June 2014)

— ***Classwork*** —

Question 1. Conic basis

The aim of this task is to find a suitable projective bases for the pencil of conics through the following four points:

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

a) In the pencil of conics through these four points there are three degenerate conics. Create a sketch of each of these.

Note: Several subtasks up to h) will be discussing these three conics. So it makes sense to start a three-column arrangement on your sheet.

b) Let $\mathcal{K}_\lambda$ for $\lambda \in \mathbb{R} \cup \{\infty\}$ be defined as the conic section

$$\mathcal{K}_\lambda := \{E \mid (A, B; C, D)_E = \lambda\}$$

Assign labels $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_\infty$ to the three conics you just sketched.

If you don't know the correct assignment immediately, *do not* look them up in your lecture notes, but instead try to work them out based on the definition of the cross ratio, which you might want to write down for this purpose.

c) Show that your assignment is correct, i.e. that every point E on each of these conic sections does result in the required cross ratio.

d) Determine homogeneous coordinates for the lines which comprise these degenerate conics.

e) As an example, express one of the three degenerate conics as a homogeneous quadratic equation in the variables x, y and z.

f) Represent each of the three degenerate conics as a matrix M_λ for $\lambda \in \{0, 1, \infty\}$.

g) In what ways may these matrices be modified such that they still represent the same conic?

h) Using such modifications, find three matrices N_λ which represent the same conics $\mathcal{K}_\lambda$ but which also satisfy the following additional requirement:

$$N_1 = N_0 + N_\infty$$

i) Give a short reason why every linear combination $\mathcal{K}_{\alpha,\beta}$ with

$$N_{\alpha,\beta} := \alpha N_\infty + \beta N_0 \qquad\qquad \alpha, \beta \in \mathbb{R}, \quad (\alpha, \beta) \neq (0, 0)$$

represents a conic through the four points A, B, C, D.

j) Formulate a conjecture how, based on the work accomplished so far, one might be able to obtain a conic $\mathcal{K}_\lambda$ for any given cross ratio λ. Right now you don't have to prove this conjecture (yet); proving it will be the goal of subsequent subtasks.

k) For each of the four given points, i.e. for $P \in \{A, B, C, D\}$, let $t_{\lambda,P}$ be the tangent to $\mathcal{K}_\lambda$ in point P. Show that the cross ratio

$$\mu(\mathcal{K}_\lambda) := (t_{0,P}, t_{\infty,P}; t_{\lambda,P}, t_{1,P})$$

does not depend on the choice of P.

l) Prove the equation

$$\mu(\mathcal{K}_{\alpha,\beta}) = \frac{\alpha}{\beta}$$

for the linear combinations $\alpha N_\infty + \beta N_0$ introduced in subtask i).

m) Show that $\mu(K_\lambda) = \lambda$ holds.

Note: Consider the limiting process $E \to P$ for the point E which has been used in the definition of $\mathcal{K}_\lambda$. Choose P in a clever way.

n) Find the conic $\mathcal{K}_{-1}$ of all points which have harmonic lines to the points A through D. Try to interpret this geometrically, and to sketch it as well.

o) Now consider the general situation of four conics which intersect in exactly four points, even if these four points are not the A through D given in the problem statement above. Show that the cross ratio of the four tangents in these common points is still independent from the choice of the common point.

— ***Homework***—

Question 2. Generalization of quadrilateral sets

Given a convex n-gon with corners c_1 through c_n, together with a line l which does not intersect that n-gon. For $1 \leq i \leq n$ one can obtain points a_i by orthogonal projection of the c_i onto line l. One can also obtain points b_i as the points of intersection between the extended polygon edges $c_i \vee c_{i+1}$ and the line l. All indices in this task are to be interpreted modulo n.

In this task, you are asked to show that on l the following equation holds:

$$\prod_{i=1}^{n}[a_i, b_i] = \prod_{i=1}^{n}[a_{i+1}, b_i]$$

a) Create a sketch for $n = 5$.

b) In what sense is this task a generalization of quadrilateral sets? For which n do you get the situation which most closely resembles that of quadrilateral sets? Create a sketch of that situation as well.

c) Choose suitable coordinates, so that the points c_i can be easily interpreted as versions of a_i which have been lifted to a height h_i

d) Which triples of collinear points must result from the construction? Express these using the coordinates you just chose.

e) Reformulate the equations which express these collinearities in such a way that they contain both the height variables and determinants in homogeneous coordinates on l.

f) Combine these equations in a suitable way, to obtain the advertised final result:

$$\prod_{i=1}^{n}[a_i, b_i] = \prod_{i=1}^{n}[a_{i+1}, b_i]$$

g) Under which contraints does the reverse conclusion hold true as well? Or in other words, what extra conditions are needed to ensure that the equation implies liftability to a non-degenerate n-gon?