Technische Universität München
Zentrum Mathematik

Prof. Dr. Dr. Jürgen Richter-Gebert, Martin von Gagern

Projective Geometry SS 2014

www-m10.ma.tum.de/ProjektiveGeometrieSS14

Worksheet 4 (12. May 2014)

— ***Classwork***—

Question 1. Rational projective plane

a) Argue briefly that $\mathbb{QP}^2 = (\mathcal{P}_\mathbb{Q}, \mathcal{L}_\mathbb{Q}, \mathcal{I}_\mathbb{Q})$ is a projective plane.

b) State the transformation matrix M which expressed a rotation around the origin by an angle of $45°$ counter-clockwise.

c) The set of rational points shall now be combined with a rotated version of itself, and the result shall be supplemented to form a projective plane. The result is a structure which is a part of $\mathbb{RP}^2$, and the operations $\vee$ and $\wedge$ will be interpreted in that real plane.

$$\mathcal{P}_0 := \mathcal{P}_\mathbb{Q} \cup \{M \cdot p \mid p \in \mathcal{P}_\mathbb{Q}\} \qquad \mathcal{L}_0 := \{a \vee b \mid a, b \in \mathcal{P}_0\}$$
$$\mathcal{P}_{i+1} := \{a \wedge b \mid a, b \in \mathcal{L}_i\} \qquad \mathcal{L}_i := \{a \vee b \mid a, b \in \mathcal{P}_i\}$$
$$\mathcal{P} := \bigcup_{i=0}^{\infty} \mathcal{P}_i \qquad \mathcal{L} := \bigcup_{i=0}^{\infty} \mathcal{L}_i$$

Examine whether a finite number of iterations of the above will already lead to a fixed point. Phrased differently, whether there exists some $i < \infty$ such that $\mathcal{P}_i = \mathcal{P}_{i+1}$ und $\mathcal{L}_i = \mathcal{L}_{i+1}$.

d) Demonstrate briefly that the sets $\mathcal{P} \subseteq \mathcal{P}_\mathbb{R}$ and $\mathcal{L} \subseteq \mathcal{L}_\mathbb{R}$ constructed above will form a projective plane if taken together with the incidence relation of the real projective plane, restricted to these objects:

$$\mathcal{I} := (\mathcal{P} \times \mathcal{L}) \cap \mathcal{I}_\mathbb{R}$$

e) Try to describe this plane as a projective plane over some number field. If you find a suitable field, write it down using common notation.

f) In the lecture it has been shown that in $\mathbb{RP}^2$, every collineation is a projective transformation. Find out where that proof will fail for the plane $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ constructed above.

g) Prove or disprove the statement that every collineation is a projective transformation in the plane constructed above,

Question 2. Geometrically computable operations

For three points A, B, C on a projective line, the function h shall be defined by

$$h(A, B; C) = D \quad \Leftrightarrow \quad (A, B; C, D) = -1$$

So this function h computes a fourth point in such a way that it forms a harmonic set with the given three points. Furthermore, the points 0, 1 and ∞ shall be designated on the line of computation, as well as two more points x and y.

a) Find out which of the following points on the line of computation can be constructed from the given points by repeated application of the function h. For those where such a construction is possible, state a formula to obtain that point. For those where no such construction is possible, give a reason for this fact.

(1) $3 \cdot x$
(2) $x + y$
(3) $x - y$
(4) $x \cdot y$
(5) $\frac{1}{x}$
(6) $\frac{x}{y}$
(7) $\sqrt{x}$
(8) x^2
(9) e^x

Note: Operations which you have already reduced to applications of h can be used in subsequent definitions as a kind of shorthand notation. So you don't really have to state every single expression as a nested expression of h applications, as long as you have ensured that such a notation would be possible in theory.

b) The function h is not defined if two of the three points A, B, C coincide. Investigate which of the expressions are affected by this case, and give a case distinction which could handle the required computation in these special cases. You may assume that 0, 1 and ∞ are pairwise distinct.

c) Suppose someone were to denote yet another point, claiming it had position $\sqrt{2}$ with respect to the projective scale in question. Can you verify this claim using some incidence configuration? If not, why is this impossible? If you can verify the point, can you also use that construction to construct $\sqrt{2}$.

★d) Suppose h were not defined in terms of the cross ratio, but instead some arbitrary function operating on the points of the projective line. Still, addition and multiplication shall be defined using this. Which additional properties are required for h if the operations of addition and multiplication defined derived from it are to describe a number field, i.e. satisfy all field axioms?

— ***Homework*** —

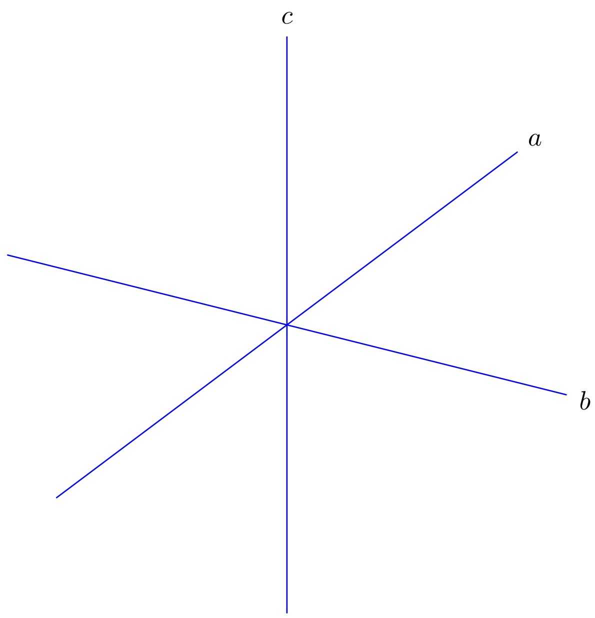
Question 3. Computations using slopes

If you only consider the slopes of the lines in the projective plane, these will form a projective line: all real numbers can occur as slopes, and in addition to these there are lines with infinite slope.

a) Create a draft explaining how to construct the point D which forms a harmonic set with three given points A, B, C, i.e. the point D such that $(A, B; C, D) = -1$. Write a step-by-step description of this construction.

b) Dualize this construction description.

c) Use this dual construction to construct the harmonic line d for the following three lines $a, b; c$.



d) Show that the cross ratio of the four lines you just constructed can also be written as a fraction of determinants of homogeneous coordinates of the lines involved.

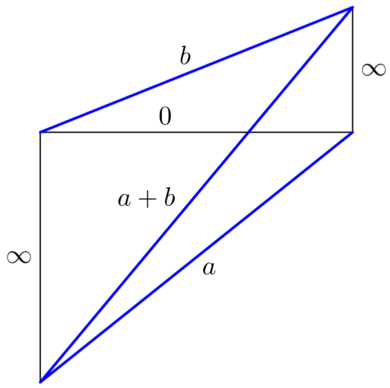
e) Consider in general the cross ratio of four arbitrary lines though a common point, computed as we just did using the homogeneous coordinates of the lines. Show that the points of intersection of these lines with the line at infinity have the same cross ratio.

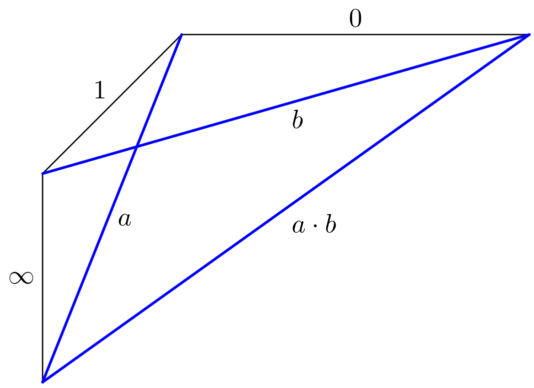
f) Intersect the four concurrent lines in your construction for sub-task c) with another (finite) line. Check, using a suitable construction, whether the four points of intersection form a harmonic set.

g) Explain how the cross ratio of four slopes could be defined in a sensible way even if the four lines in question don't pass through a common point.

h) Lines with slopes 0, 1 and ∞ fix a projective scale. Check that with respect to this scale, measuring slopes via the coordinate vector of a line leads to the same result as computing the slope via the cross ratio relative to the basis stated above.

i) Prove that the following constructions can actually be used to add and multiply slopes of lines, the way the labels suggest. You may use affine or Euclidean arguments.





Question 4. The whole plane

Define the plane $\mathbb{ZP}^2$ in the common way, identifying scalar multiples in $\mathbb{Z}^3 \setminus \{0\}$ and testing incidences using the scalar product.

a) Is the structure defined above in fact a projective plane?

b) Is there a projective plane over some field which is isomorphic to the structure above? If so, name the underlying field. If not, prove that no such field can exist.

c) Building on other tasks from this sheet, find an incidence configuration which cannot be realized as a part of the above structure, but can be realized in $\mathbb{RP}^2$.