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Projective Geometry SS 2014
www-m10.ma.tum.de/ProjektiveGeometrieSS14
Worksheet 3 (5. May 2014)

— ***Classwork***—

Question 1. Desargues' Theorem

a) Create two sketches of Desargues' theorem: one in it sprojective version, as general as possible, and the other in a Euclidean setting where the final conclusion is an incidence with the line at infinity.

b) Formulate both variants in words of formulas.

c) Prove the theorem using theorems known from school.

Question 2. A theorem for parallels

The following theorem (called *Scherensatz* in German, which would translate as *scissor theorem*, but I don't know whether that's an established name) states:

If the following incidences are satisfied:

- a, b, A, B lie on a common line
- c, d, C, D lie on a common line
- $a \vee c$ and $A \vee C$ are parallel
- $a \vee d$ and $A \vee D$ are parallel
- $b \vee c$ and $B \vee C$ are parallel

then the lines $b \vee d$ and $B \vee D$ are laralle as well.

a) Create sketch of this theorem.

b) Use Desargues' theorem to prove this theorem. Make sure you prove any special cases which might arise.

c) Formulate a projective generalization of this theorem, and draw it in such a way that all points and lines of the configuration are finite and visible in the construction.

Question 3. Pappos in coordinates

In this task, Pappos' theorem shall be proven using explicit coordinates and some real variables. You may disregard degenerate situations unless stated othwerwise.

a) How many points in general position can be choosen arbitrarily in Pappos theorem in such a way that the other points will follow from the chosen ones?

b) Show that any projective transformation will map one instance of the configuration of Pappos' theorem again to an instance of that theorem. How can this fact be used to your advantage in the remainder of the proof?

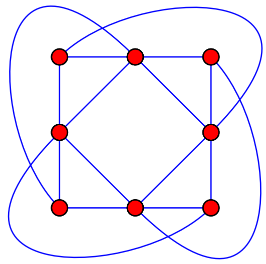
c) Consider two Pappos configurations as equivalent if one is the image of the other under some projective transformation. How many real parameters are required to uniquely characterize each equivalence class?

d) Use these real parameters which you just counted as variables, and formulate coordinates for all the points of the configuration depending on this.

e) Formulate the conclusion of the theorem using these coordinates, and show that it actually holds.

f) Examine which combinations of the parameters described above will cause Pappos' theorem do degenerate, and consider how this affects the proof.

Question 4. Embedding a 8_3 configuration



In this task, the 8_3 configuration depicted on the right shall be embedded into a projective plane $\mathbb{KP}^2$ using explicit coordinates. So the points and lines of the configuration should be actial points and lines of the containing projective plane, and incidence between elements of the configuration shall be the same as the incidence relation of the plane, restricted to the set of points and lines included in the configuration.
You may restrict your considerations to the non-degenerate case, where no two points or lines of the configuration will coincide.

a) How many scalar parameters are required to describe such a 8_3 configuration uniquely up to projective transformation?

b) Compute explicit coordinates for the points of the configuration. Do so in such a way that you obtain as many "simple" coordinates as possible, and only scalar variables for the parameters you just counted.

c) Describe which incidences of the configuration have to be satisfied, in addition to those already guaranteed by your choice of coordinates.

d) Express these incidences using determinants.

e) Show that this 8_3 configuration can never be embedded into $\mathbb{RP}^2$ but embeds just fine into $\mathbb{CP}^2$.

f) Embeddings of this configuration into $\mathbb{CP}^2$ fall into different equivalence classes, where elements of a single class only differ by a projective transformation. How many such equivalence classes are there?

Question 5. Cross ratios on the line and in the plane

For the scope of this task, the cross ratio of four collinear points shall be defined as follows:

(i) For the points

$$A = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad B = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad C = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad D_\lambda = \begin{pmatrix}\lambda\\0\\1\end{pmatrix}$$

the cross ratio $(A, B; C, D)$ (for arbitrary λ) shall be defined to be

$$(A, B; C, D_\lambda) := \lambda$$

(ii) The cross raio of four arbitrary collinear points in the plane is invariant under projective transformations of the plane.

a) Argue that the above definition is sufficient to uniquely define the cross ratio of any four points in the plane.

b) The cross ratio of four points, as seen from a fifth point O, is defined just as it is in the lecture:

$$(A, B; C, D)_O := \frac{[OAC][OBD]}{[OAD][OBC]}$$

Demonstrate that ifA, B, C, D lie on a common line, and O does not lie on said line, then the following holds:

$$(A, B; C, D) = (A, B; C, D)_O$$

Note: This is a generalization of the proof which was only done over $\mathbb{R}$ in the lecture. Where the lecture used lengths as the basis of its definition, this one here uses coordinates instead.