Technische Universität München
Zentrum Mathematik

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Projective Geometry SS 2014

www-m10.ma.tum.de/ProjektiveGeometrieSS14

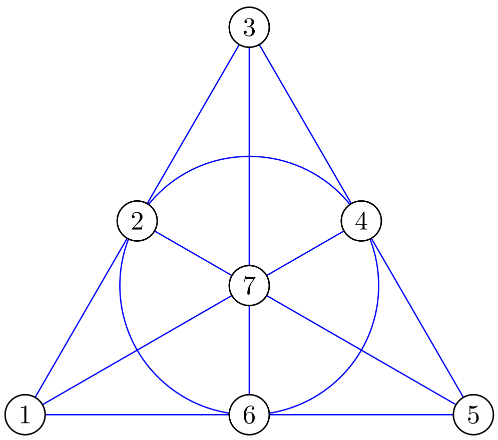
Worksheet 2 (24. April 2014)

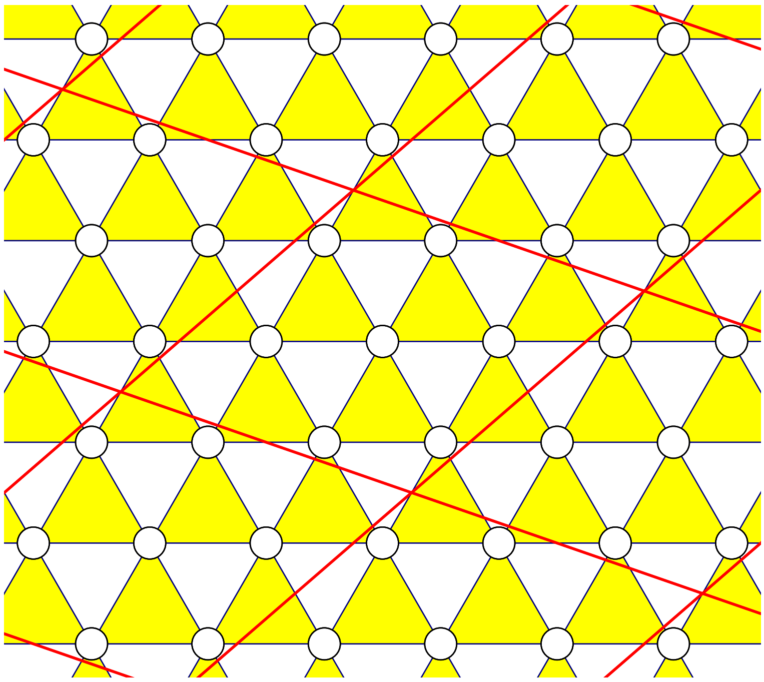
— ***Classwork***—

Question 1. Symmetries of the Fano plane

Consider the Fano plane consisting of the lines

$$(1, 2, 3), (3, 4, 5), (5, 6, 1), (1, 4, 7),$$
$$(3, 6, 7), (5, 2, 7), (2, 4, 6)$$





Imagine the depicted triangular grid to extend infinitely in all directions.

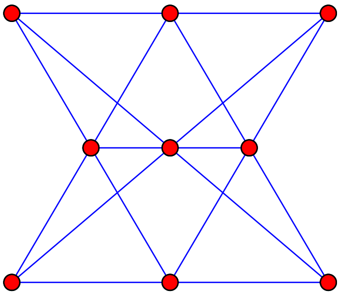
a) Enter the numbers $1 \dots 7$ into the circles of the triangular grid in such a way that every yellow triangle corresponds to one line of the Fano plane. Furthermore, all diamonds should contain the same pattern of numbers, so they are related by a translation. Every number should appear in every diamond exactly once.

b) Describe several automorphisms τ of the Fano plane which satisfy the equation $\tau^7 = \mathrm{id}$. Also describe several automorphisms satisfying $\tau^3 = \mathrm{id}$.

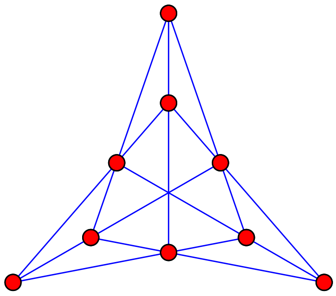
c) What is the cardinality of the group of automorphisms of the Fano plane? Can you explain this number using the structure of the triangular grid? In order to answer this question, recall the arbitrary choices you made while filling in the triangular grid.

Question 2. 9_3 configurations

A (n_r, n_r) configuration is often simply called a n_r configuration. Here you see several drawings of 9_3 configurations.



(1)



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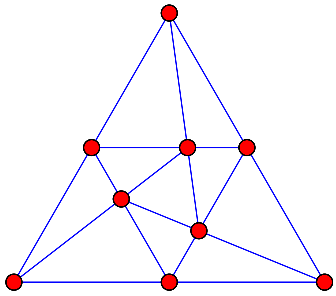

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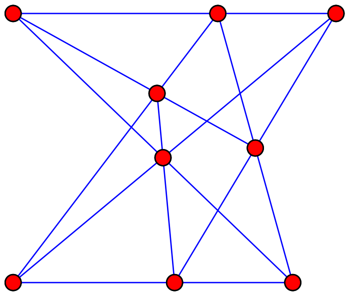

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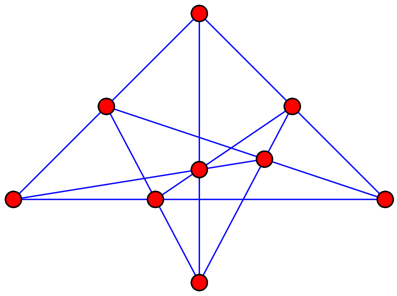
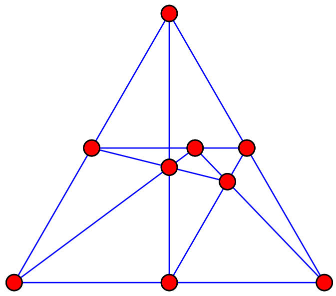


(5)



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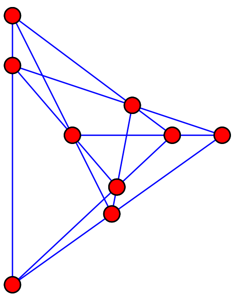

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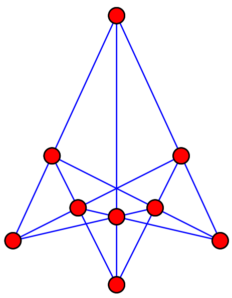

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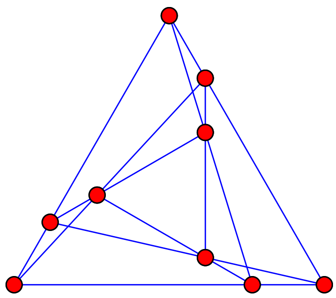

(9)

a) Identify which of these images represent isomorphic incidence structures. Label the points in such a way that for isomorphic configurations, the labels of all collinear triples of points agree, so that the isomorphism is reafily apparent. Most of the subsequent subtasks can be addressed for all isomorphic configurations at once.

b) Choose one of the given configurations which is *not* isomorphic to Pappos' theorem. Try to reproduce this drawing (without measuring the original). Which parameters can be chosen arbitrarily during that process?

c) Consider how you could *compute* coordinates for the points in the drawing you just created (or at least tried to create).

d) For every configuration, determine the dual incidence structure, and decide which of the depicted primal configurations are isomorphic to that. Support this claimed isomorphism by labeling the lines of the original in a suitable way.

★e) Construct additional 9_3 configurations and investigate, which of the depicted configurations are isomporphic to yours.

★f) Compute coordinates for every isomorphism class. Keep the computation as general as possible, so wherever you can make an arbitrary choice, introduce a new variable for that and express dependent coordinates in terms of these variables.

— ***Classwork for groups 1 and 2, homework for groups 3 and 4*** —

Question 3. Finitely Pappos

a) What is the smalles finite projective plane in which a non-degenerate instance of Pappos' theorem can be realized, and in which this theorem is generally true?

b) Create a sketch of the corresponding projective plane, and in that emphasize one instance of the theorem.

c) Also create a sketch of this theorem in $\mathbb{RP}^2$. Label the points in both sketches correspondingly, so that the common combinatorics become clear.

— ***Homework*** —

Question 4. Collineations

Given the following points in $\mathbb{CP}^2$:

$$A = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad B = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad C = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad D = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad D' = \begin{pmatrix}1\\i\\1\end{pmatrix}$$

a) Find a projective transformation, which fixes the points A, B and C and maps D onto D'. Write down a function $\tau_a : \mathcal{P} \to \mathcal{P}$ which describes the effect of this transformation for the points of the plane.

b) Find a collineation which is not a projective transformation but still maps the points as stated above. Describe this second operation using a function $\tau_b : \mathcal{P} \to \mathcal{P}$

c) Consider both the collineations you obtained in the previous subtasks, and apply each of them to the point $E = (i, 0, 1)^T$. Compare the results.

d) Demonstrate that the map τ_b is well defined.

Question 5. Moulton plane

The so-called Moulton plane is an affine plane, in which Desargues' theorem does not hold everywhere. Its projective closure can be described as follows:

$$\mathcal{P} = \mathcal{P}_{\mathbb{R}} \qquad \mathcal{L} = \mathcal{L}_{\mathbb{R}} \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} \mathcal{I} \begin{pmatrix}a\\b\\c\end{pmatrix} \Leftrightarrow \begin{cases} 2ax + by + cz = 0 & \text{if } x \cdot z > 0 \text{ and } a \cdot b > 0 \\ ax + by + cz = 0 & \text{otherwise} \end{cases}$$

a) Verify, that the special case in the incidence relation $\mathcal{I}$ only applies to points to the right of the y axis and lines with finite negative slope.

b) Show that the incidence relation is well defined.

c) Create a descriptive picture of this plane.

d) Describe a process by which joining lines and points of intersection can be computed.

e) Verify that this projective Moulton plane satisfies all the axioms of a projective plane.

f) Create a construction which demonstrates that Pappos' theorem is not generally true in this plane.

g) Create a construction which demonstrates that Desargues' theorem is not generally true in this plane.