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Projective Geometry SS 2014

`www-m10.ma.tum.de/ProjektiveGeometrieSS14`

Worksheet 1 (14. April 2014)

— ***Classwork for groups 1 and 2, homework for groups 3 and 4*** —

Question 1. Incidence matrix with gaps

★a) Fill in the gaps in the following matrix in such a way that it represents the cincidence structure of a finite projective plane. The symbol ✔ denotes an icidence, the symbol – the absence of an incidence.

	A	B	C	D	E	F	G	H	I	K	L	M	N
a								–		✔			
b			–	–				–		–	–	–	✔
c			–						✔				
d			–		✔			–					
e	✔								✔				✔
f													
g			–					–					
h	✔	✔											
i							✔					✔	
k	✔				–	–				–		–	
l	✔			✔				✔					
m			✔							✔			
n					–	–		✔					

b) How can one derive a Cayley table (i.e. a table with operands as labels of the rows and columns, and the corresponding result as the corresponding table entry) for the operations $\vee$ (join) and $\wedge$ (meet)? Outline such a table and fill in some entries as examples.

c) What's the order of this finite plane?

d) Create an (unlabeled) draft of this plane.
Hint: All projective planes of this order are isomorphic.

e) Label the created draft in accordance with the incidence matrix. How unique is this labeling?

f) How many different ways are there to label the rows and columns of the matrix using homogeneous coordinates?

★g) Give one such labeling explicitely.

— ***Classwork***—

Question 2. (n_r, m_k) configurations

Given an incidence structure with points $\mathcal{P}$, lines $\mathcal{L}$ and incidence relation $\mathcal{I}$ satisfying the following conditions:

(i) For two points $p, q \in \mathcal{P}$ with $p \neq q$ there exists at most one line $l \in \mathcal{L}$ with $p\mathcal{I}l$ and $q\mathcal{I}l$.

(ii) For two lines $l, m \in \mathcal{L}$ with $l \neq m$ there exists at most one point $p \in P$ with $p\mathcal{I}l$ and $p\mathcal{I}m$.

An (n_r, m_k) configuration is an incidence structure in the sense just stated which has n points and m lines, with k points on every line and r lines through every point.

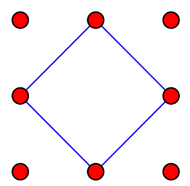
a) How do the axioms stated above differ from those of a projective plane?

b) Show that for every (n_r, m_k) configuration the equation $n \cdot r = m \cdot k$ holds.

c) Consider m lines in the real plane in general position, together with all the points of intersection between these lines. Which configuration (i.e. which n, r, k) does one obtain from this?

d) Consider n points in the real plane in general position, together with all the lines spanned by these points. Which configuration does one obtain from this?

e) Complete the following sketch to form an $(8_3, 8_3)$ configuration.



f) Find a $(13_4, 13_4)$ configuration.

★g) Find additional (n_r, m_k) configurations.

— ***Homework***—

Question 3. Counting

The number of points and lines in a projective plane over a field with n elements is equal to

$$n^2 + n + 1 = \frac{n^3 - 1}{n - 1}$$

a) Show that the above equation is true for any n.

b) Give different arguments explaining that count, one for each side of the above equation.

c) Generalize this equation for arbitrary dimension d, namely for the projective space $\mathbb{KP}^d$ over some field $\mathbb{K}$ with $|\mathbb{K}| = n$.

Question 4. Subset configurations

Given a set A with a elements. Consider all subsets with three elements as points, and all subsets with two elements as lines. Incidence shall be defined using containment of sets.

$$\mathcal{P} := \{p \subset A \mid |p| = 3\} \qquad \mathcal{L} := \{l \subset A \mid |l| = 2\} \qquad p\mathcal{I}l := l \subset p$$

a) What configuration will result for $a = 3$? Create a sketch of this configuration and label it.

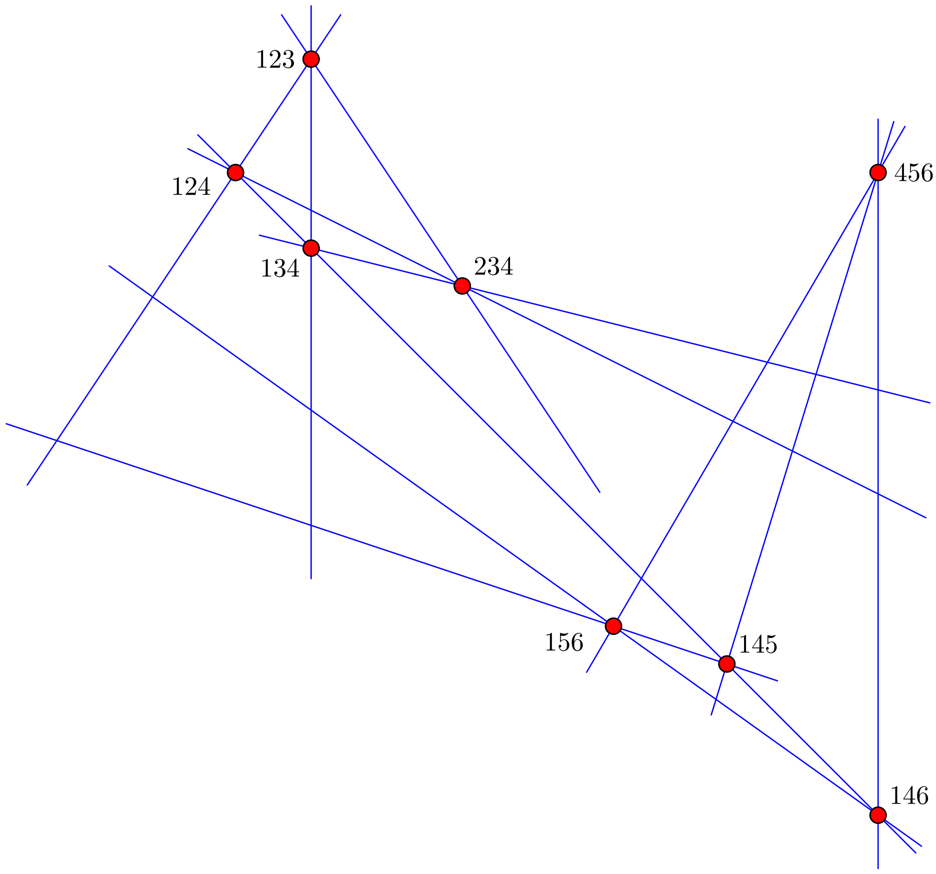
b) Also give the (n_r, m_k) name which results from $a = 4$, and create a labeled sketch of that as well.

c) What configuration will result from $a = 5$? What is special about this situation?

d) Can you draw the configuration for $a = 5$ in such a way that the lines and points of that configuration are indeed lines and points of the Euclidean plane?

e) What is the connection between the configuration for $a = 5$ and Desargues' theorem?

f) complete and label the following drawing in such a way that it reflects the configuration for $A = \{1, 2, 3, 4, 5, 6\}$.



Hint: The points of this drawing can be transferred to 5mm graph paper.

g) Give the name of the configuration which results from the general case, i.e. for arbitrary a.

h) Describe these configurations as projections of objects in higher dimensions.

Question 5. Axiomatics

An incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a projective plane if and only if the following axioms hold:

(i) For $p, q \in \mathcal{P}, p \neq q$ there exists exactly one $l \in \mathcal{L}$ with $p\mathcal{I}l$ and $q\mathcal{I}l$

(ii) For $l, m \in \mathcal{L}, l \neq m$ there exists exactly one $p \in \mathcal{P}$ with $p\mathcal{I}l$ and $p\mathcal{I}m$

(iii) There exists $a, b, c, d \in \mathcal{P}$ such that no three of these are incident to the same line.

Show using these axioms exclusively that the following statements hold in any projective plane:

a) For every point $p \in \mathcal{P}$ there exists a line $l \in \mathcal{L}$ which is not incident to p.

b) For two lines $l, m \in \mathcal{L}$ the sets of incident points have equal cardinality:

$$|\{p \in \mathcal{P} \mid p\mathcal{I}l\}| = |\{q \in \mathcal{P} \mid q\mathcal{I}m\}|$$

Hint: The lecture in this case showed two inequalities, with the proof of the second one only suggested by a claim that it is analogous to the first. Together these two inequalities result in equality only for finite sets. To properly handle infinite sets, one should define a bijection between them to compare their cardinality.

c) For finite projective planes, the number of points and the number of lines is always equal.

Hint: Several of these statements have been shown in this or a similar form in the lecture. Simply copying the proofs from the lecture is not the idea behind this task. Instead you should recall all the relevant arguments and formalize the proofs in such a way that you can best follow them yourself.