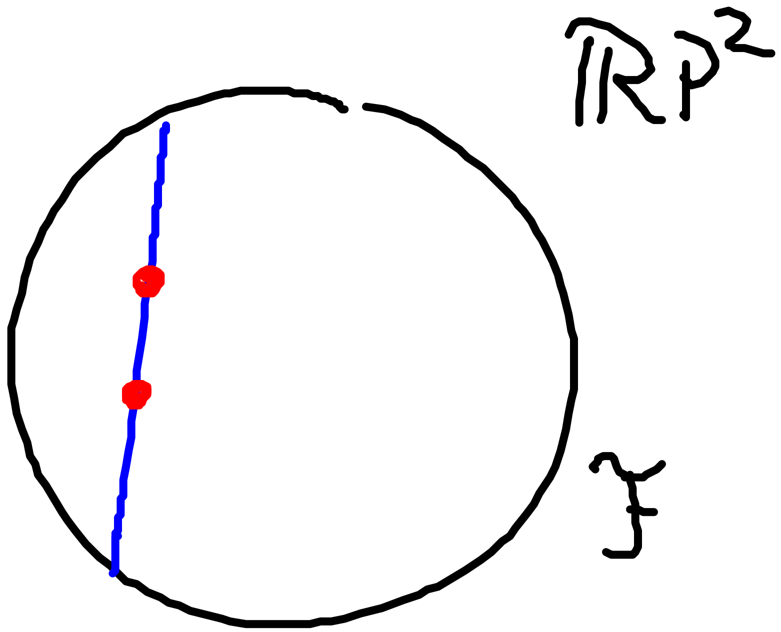
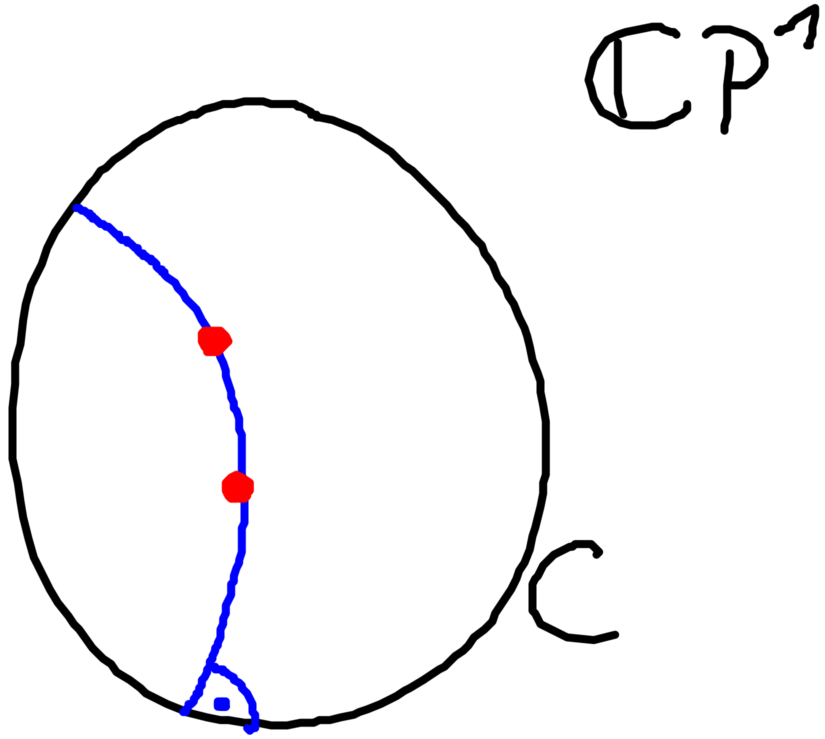
Two models for hyperbolic geometry:

Beltrami Klein



Hyperbolic Trafos are those projective $\mathbb{R}P^2$ Trafos that leave $\mathcal{J}$ invariant

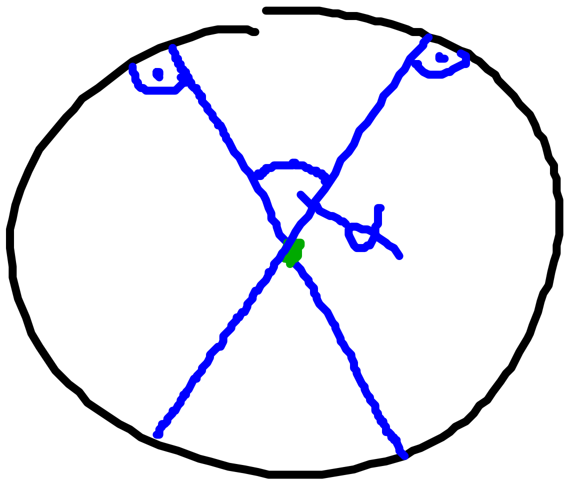
Poincaré Model



Hyperbolic Trafos are all harmonic Trafos of $\mathbb{C}P^1$ leave C invariant

- möbius Trafos
- anti-möbius Trafos

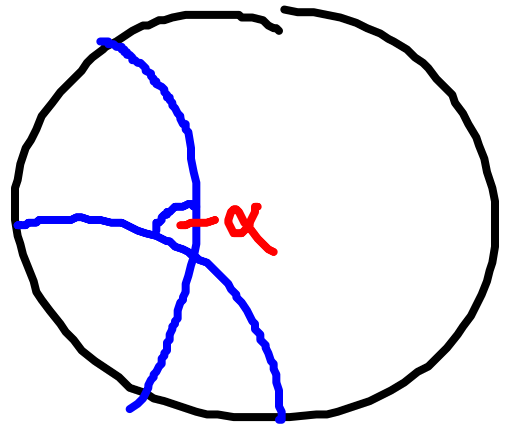
Measurements in Poincaré model



Observation:
Lines through the center of the disk are identical in both models

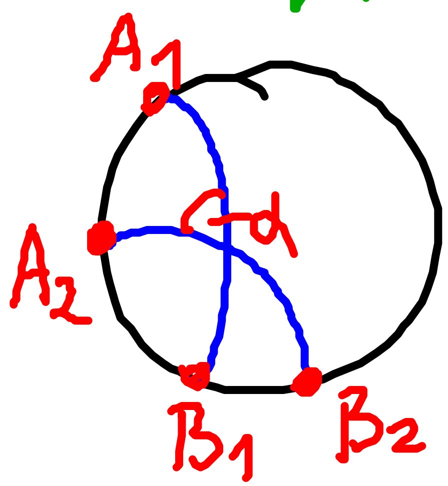
Angles between two such lines are just Euclidean Angles

Möbius and anti möbius Trafos preserve angles between cutting circles



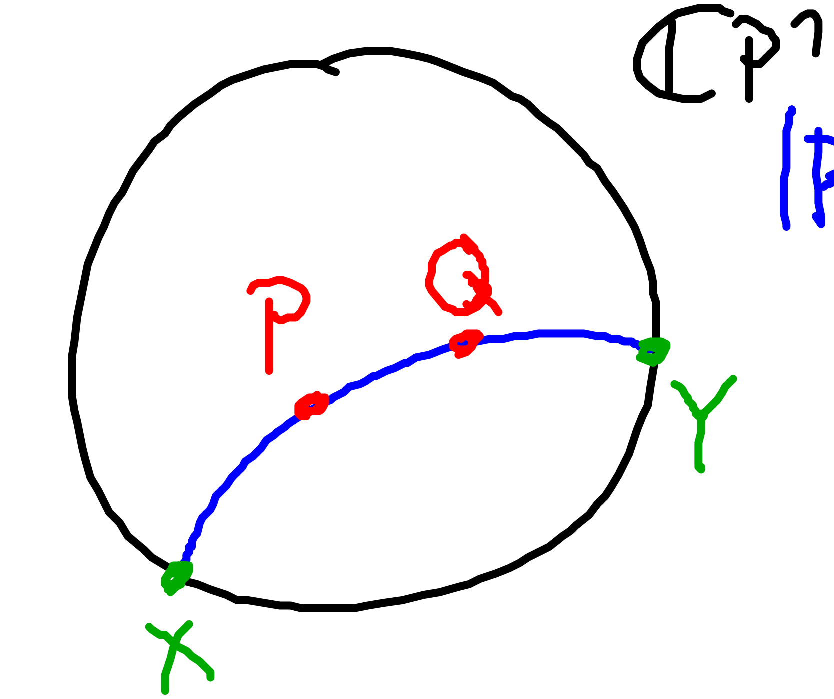
As a consequence Angles can be measured by taking angles between circles

Remark:



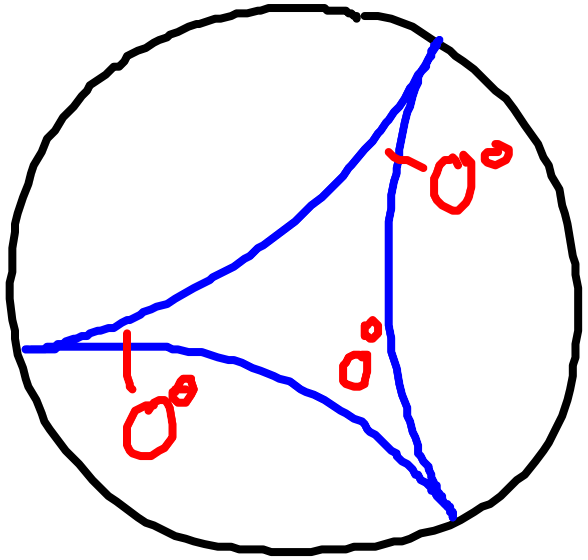
$$\alpha = \arctan\left(\sqrt{-(A_2B_2;A_1B_1)}\right)$$

Measurement of Distances in Poincaré model

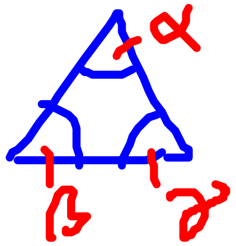


$$|P,Q| = \left| \ln \, (P,Q;X\,Y)_{\mathbb{C}P^1} \right|$$

Area of Triangles in hyperbolic Geometry



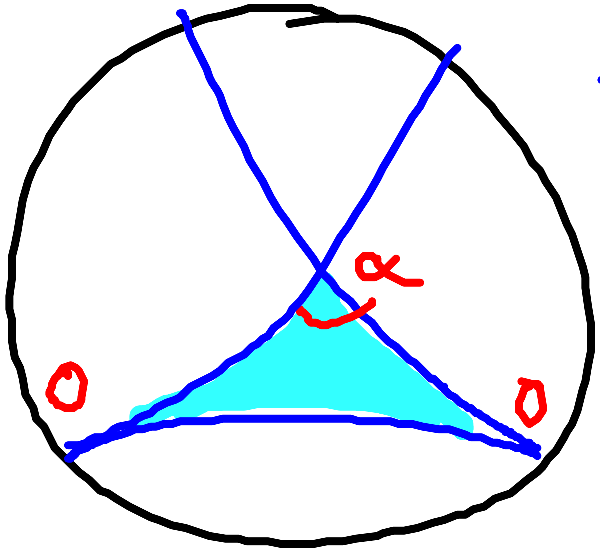
$\Delta_{\alpha,\beta,\gamma}$ triangle



The largest Triangle

Def:
$$\mathrm{vol}(\Delta_{0,0,0}) := \pi$$

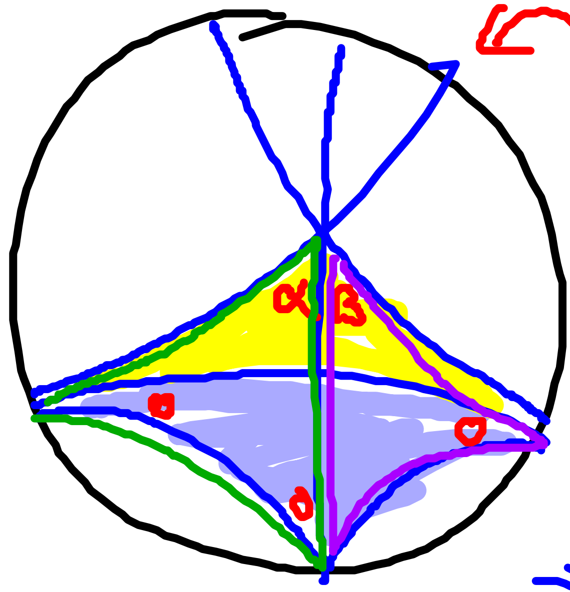
Areas should be additive invariant under Motion



$\Delta_{\alpha,0,0}$

Claim:
$$\mathrm{vol}(\Delta_{\alpha,0,0}) = \pi - \alpha$$

$$\mathrm{vol}(\Delta_{\alpha,0,0}) =: f(\alpha)$$



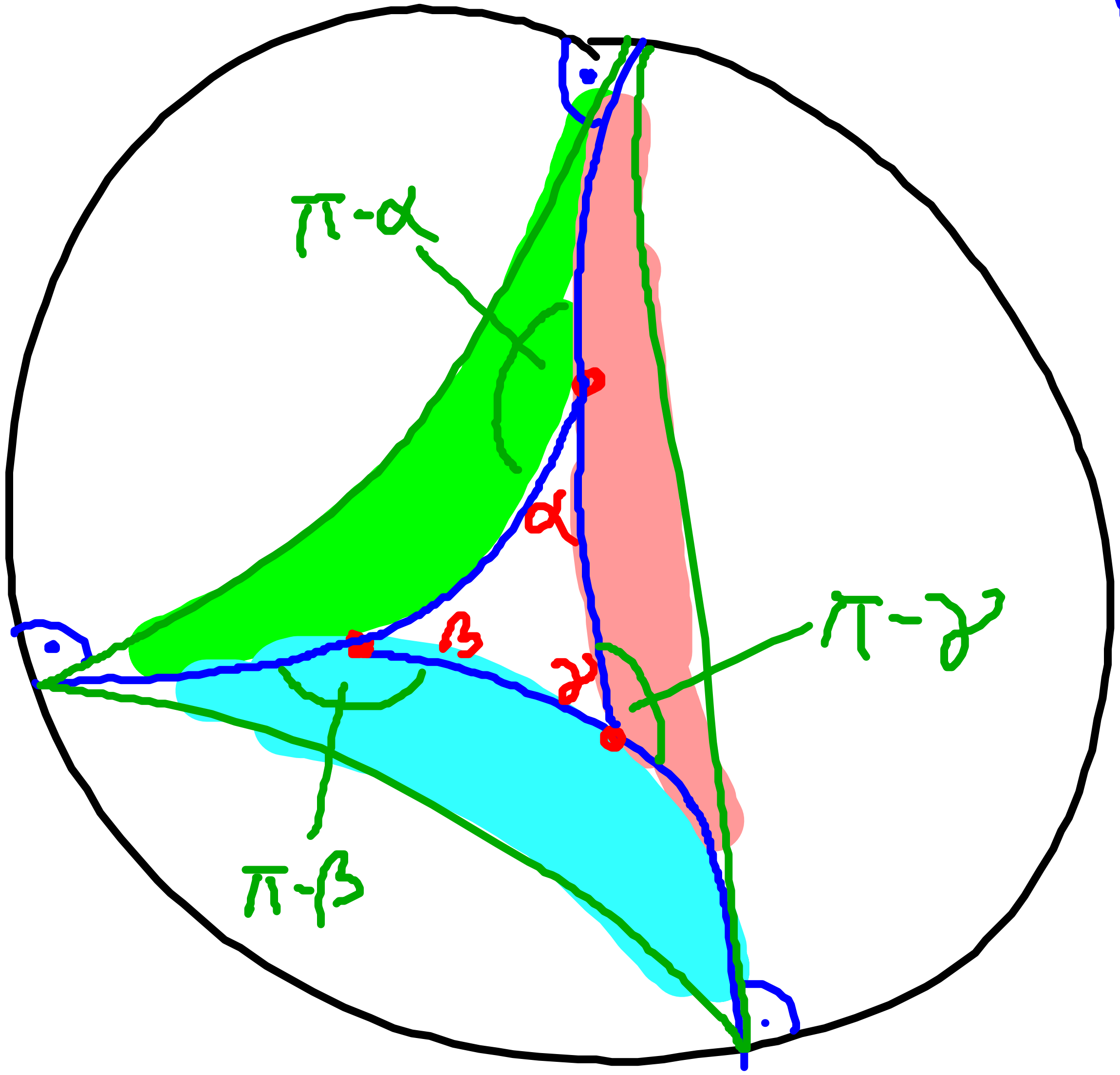
$$f(\alpha+\beta) = \boxed{f(\alpha)} + \boxed{f(\beta)} - \pi$$

Take derivatives
$$f'(\alpha+\beta) = f'(\alpha)$$

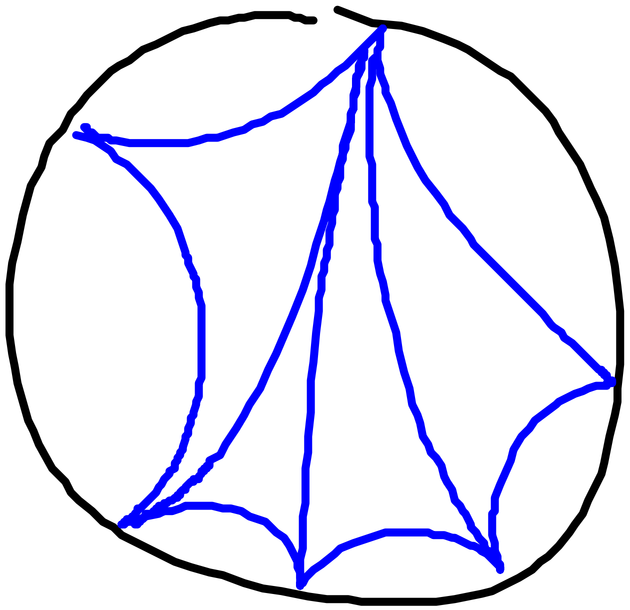
$\Rightarrow f'(\alpha)$ is constant

$\Rightarrow f(\alpha)$ is linear

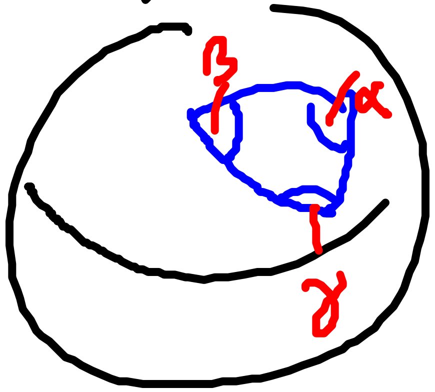
$f(0) = \pi,\ f(\pi) = 0 \Rightarrow f(\alpha) = \pi - \alpha$



$$\mathrm{vol}(\Delta_{\alpha,\beta,\gamma}) = \boxed{\pi} - \mathrm{vol}(\Delta_{\pi-\alpha,0,0}) - \mathrm{vol}(\Delta_{\pi-\beta,0,0}) - \mathrm{vol}(\Delta_{\pi-\gamma,0,0}) = \boxed{\pi - \alpha - \beta - \gamma}$$



On a sphere we have



$$\mathrm{vol}(\Delta_{\alpha,\beta,\gamma}) = \pi + \alpha + \beta + \gamma$$