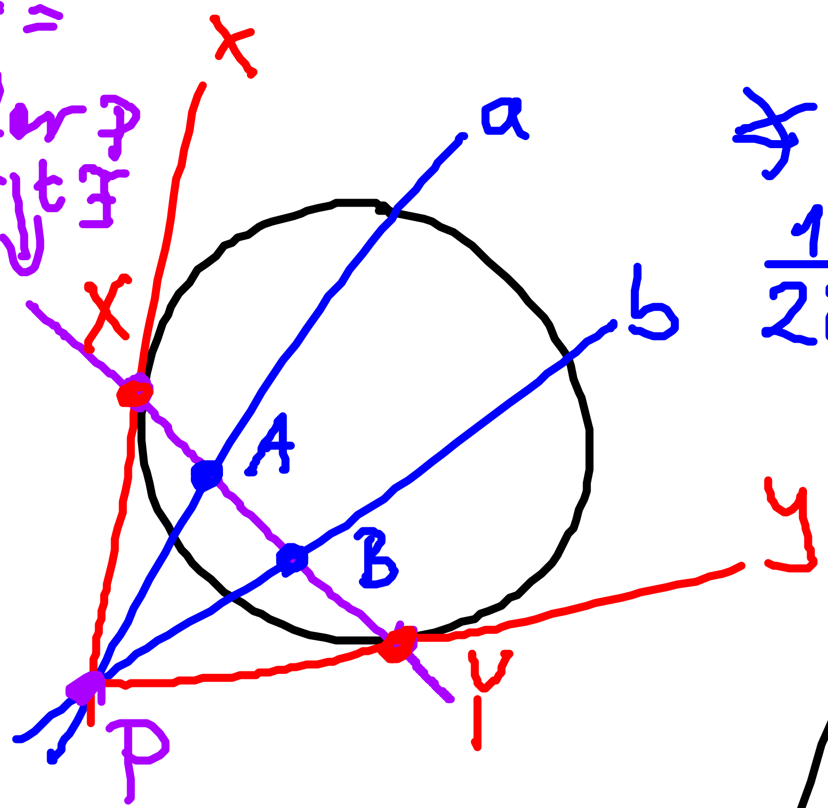
# Hyperbolic Elementary Geometry: $\mathcal{F} \sim x^2+y^2-z^2=0$

$P^* =$ Polar $P$ w.r.t $\mathcal{F}$




$$\measuredangle_{a,b} := \frac{1}{2i}\ln(a,b;x,y)$$

$$= \frac{1}{2i}\ln(A,B;X,Y)$$

$$\measuredangle_{hyp}(a,b)$$

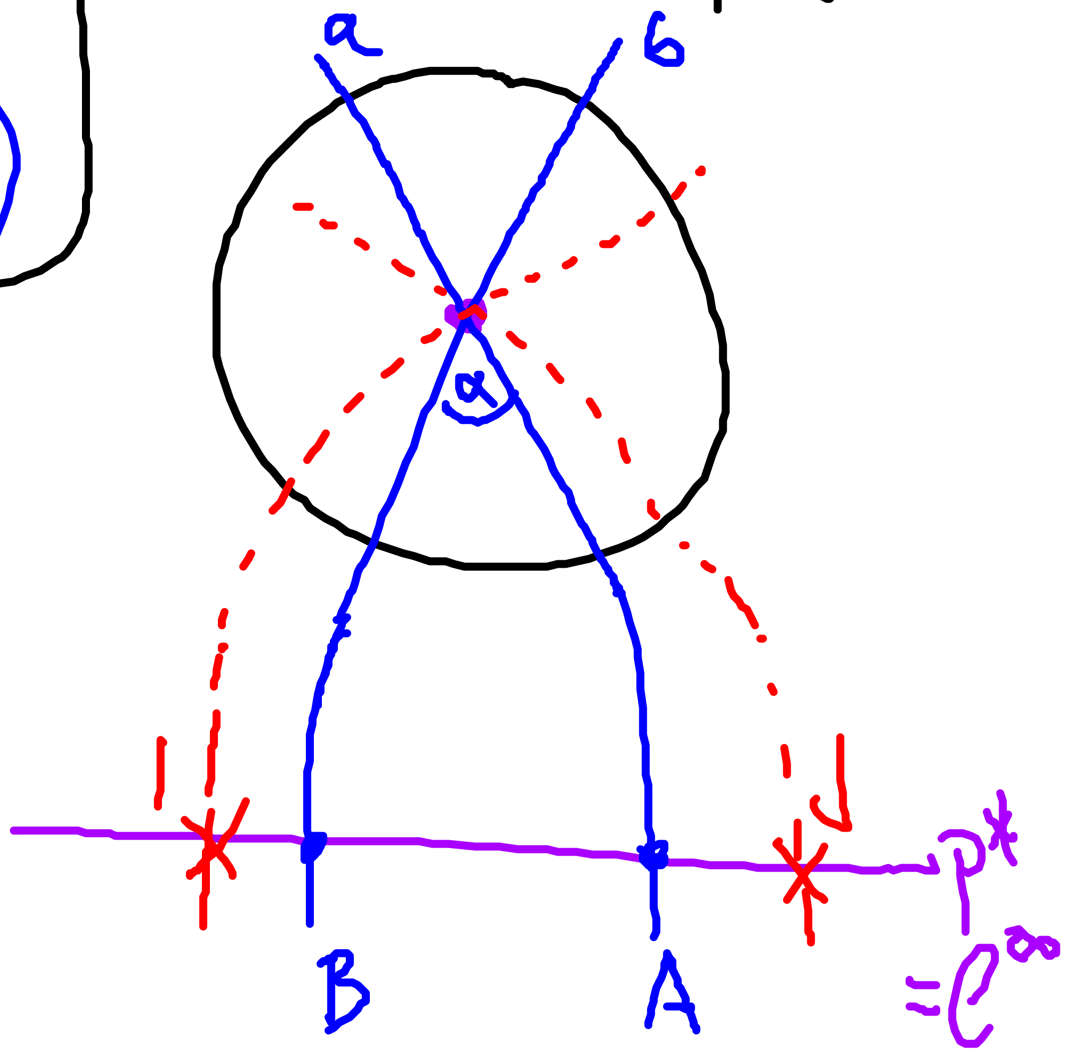
$$= \frac{1}{2i}\ln(A,B,I,J)$$

$$= \measuredangle_{euc}(a,b)$$

$P^* = A \cdot P$

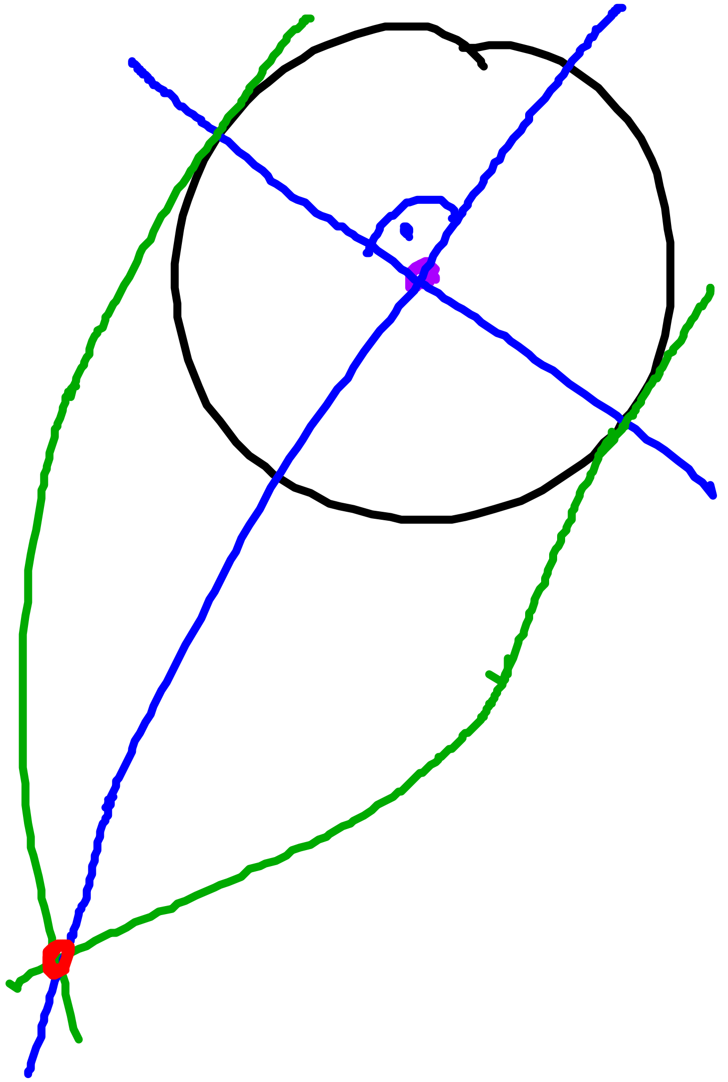
Primal Matrix of $\mathcal{F}$

Angle Measurement of two lines through the center of $\mathcal{F}$

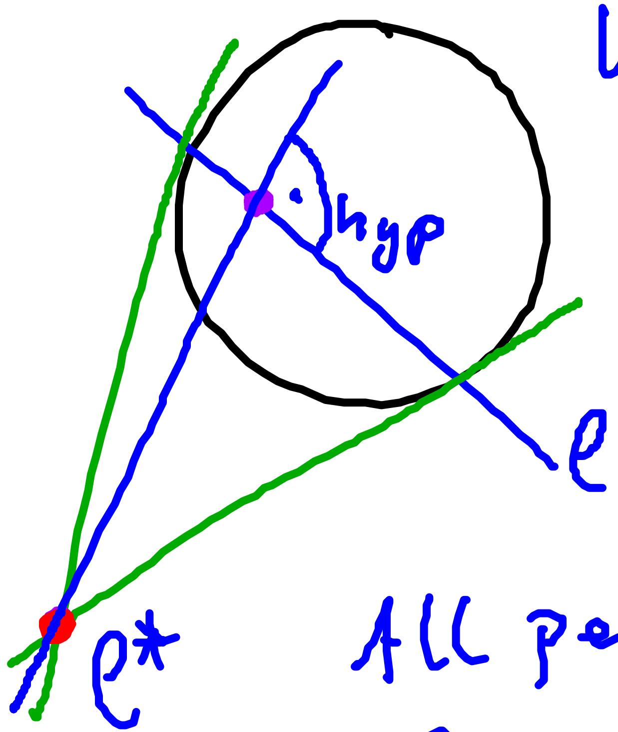


# Perpendicularity in hyp. Geometry

Situation of lines that meet in the center



general situation:




Konstruction of a perpendicular

Polar of $l$:

$$l^* = B \cdot l$$

All perpendiculars to $l$ go through $l^*$

General Formula for Perpendicular

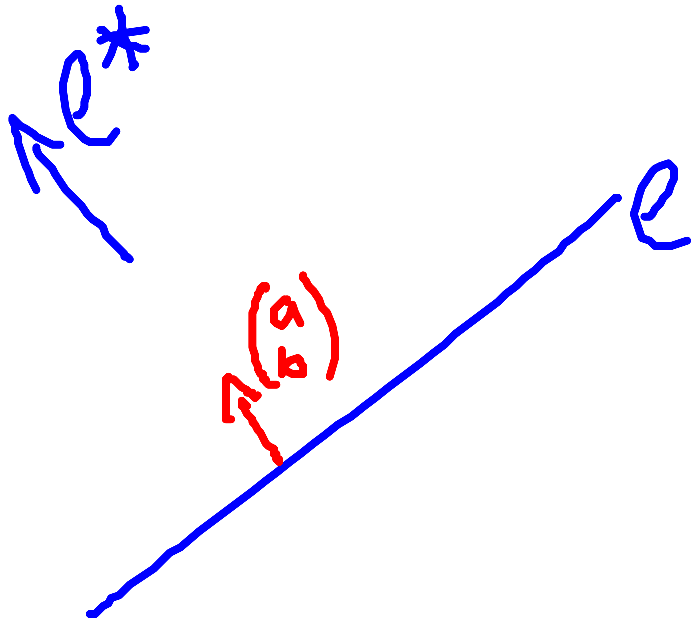
$$B \cdot l \times p$$

Same Formula for, euclidean, elliptic geometry

Example: euclidean:

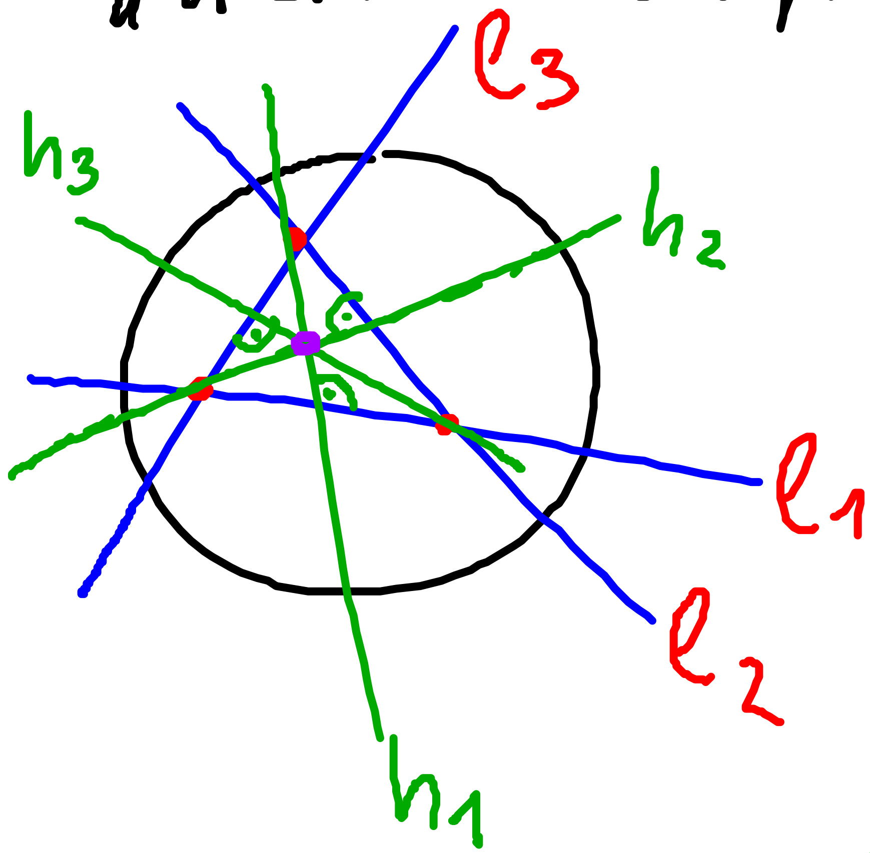
$$B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad \ell = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$ax + by + c = 0$$




$$B \cdot \ell = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \ell^*$$

• Altitudes of a triangle meet in a point




$$h_1 = (e_2 \times e_3) \times B \cdot e_1$$

$$= \langle e_2, Be_1\rangle e_3 - \langle e_3, Be_1\rangle e_2$$

$$h_2 = \langle e_3, Be_2\rangle e_1 - \langle e_1, Be_2\rangle e_3$$

$$h_3 = \langle e_1, Be_3\rangle e_2 - \langle e_2, Be_3\rangle e_1$$

$$\det(h_1, h_2, h_3) = \det(\langle e_2, Be_1\rangle e_3, \langle e_3, Be_2\rangle e_1, \langle e_1, Be_3\rangle e_2)$$

$$- \det(\langle e_3, Be_1\rangle e_2, \langle e_1, Be_2\rangle e_3, \langle e_2, Be_3\rangle e_1)$$

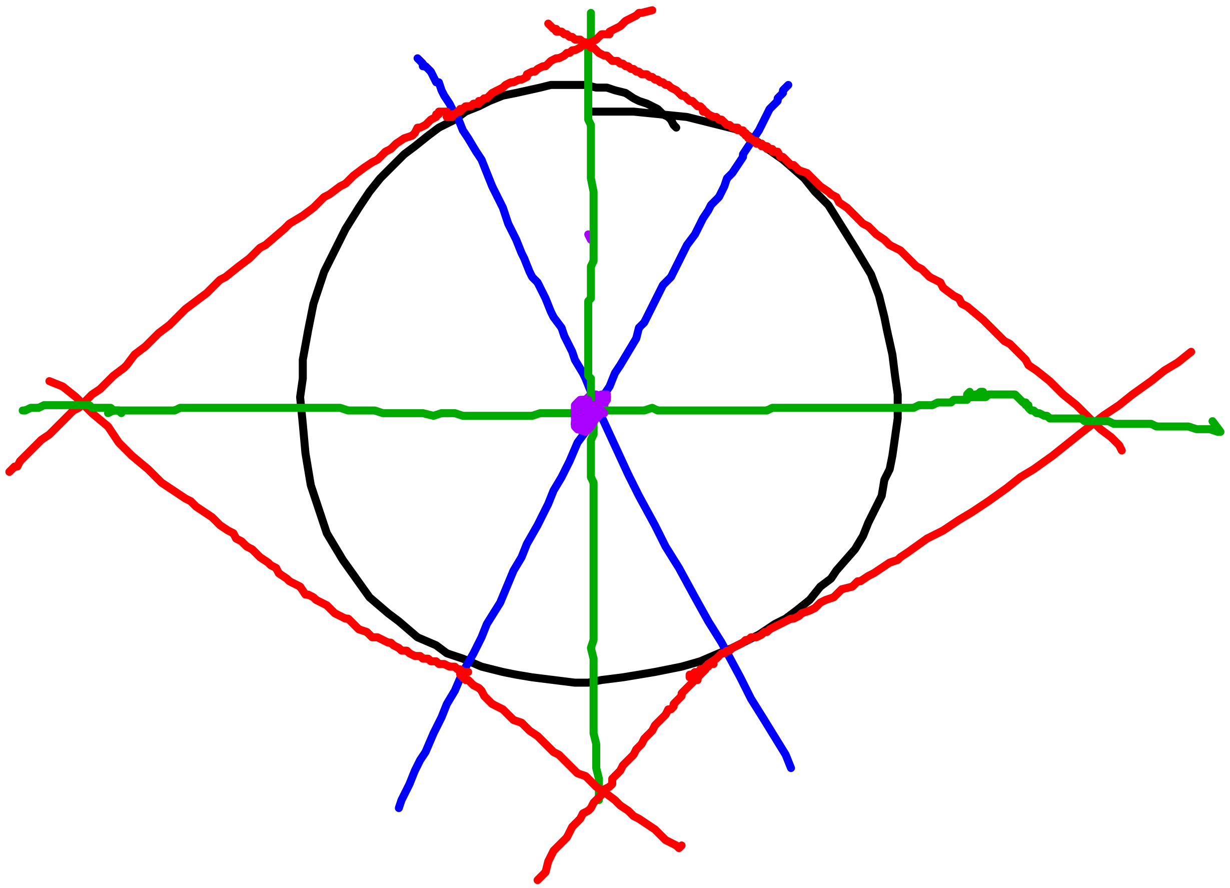
$$= 0$$

q.e.d

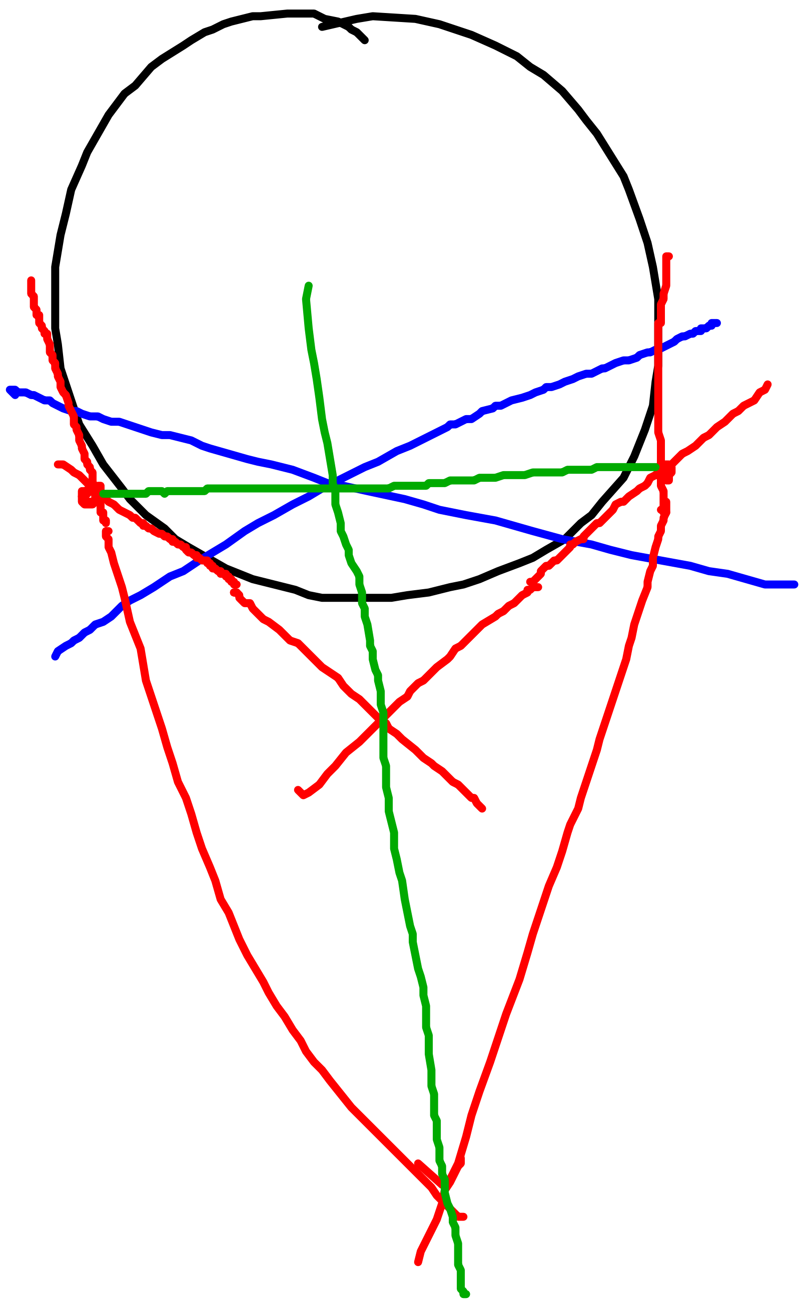
$$\langle e_1, Be_3\rangle = e_1^T B e_3 = e_1^T B^T e_3 = \langle Be_1, e_3\rangle$$

# Angle Bisections in a triangle

Situation for lines Meeting in the center



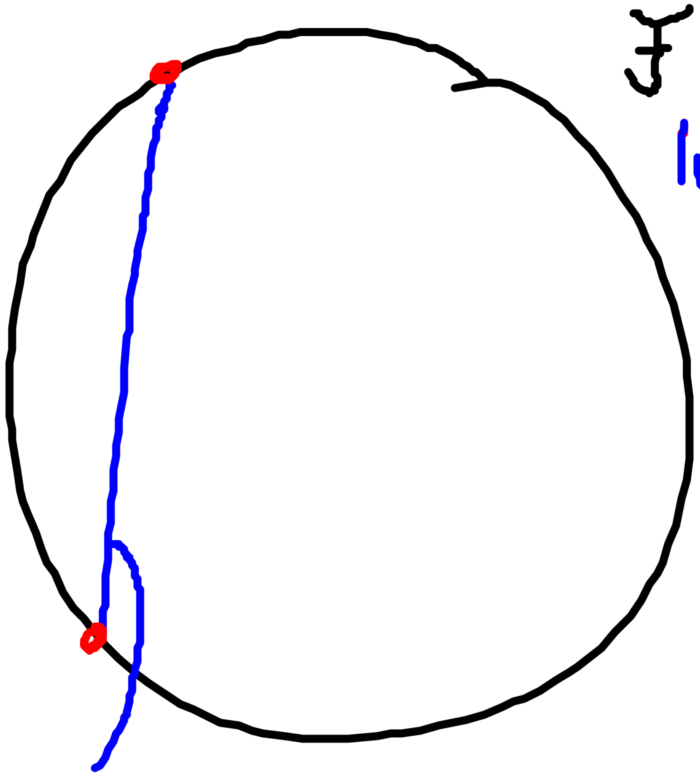
General Situation:



# Poincaré Circle Model

## Beltrami Klein Model

~ 1868-1871



$\mathcal{F}$

Interior of a Circle in $\mathbb{RP}^2$

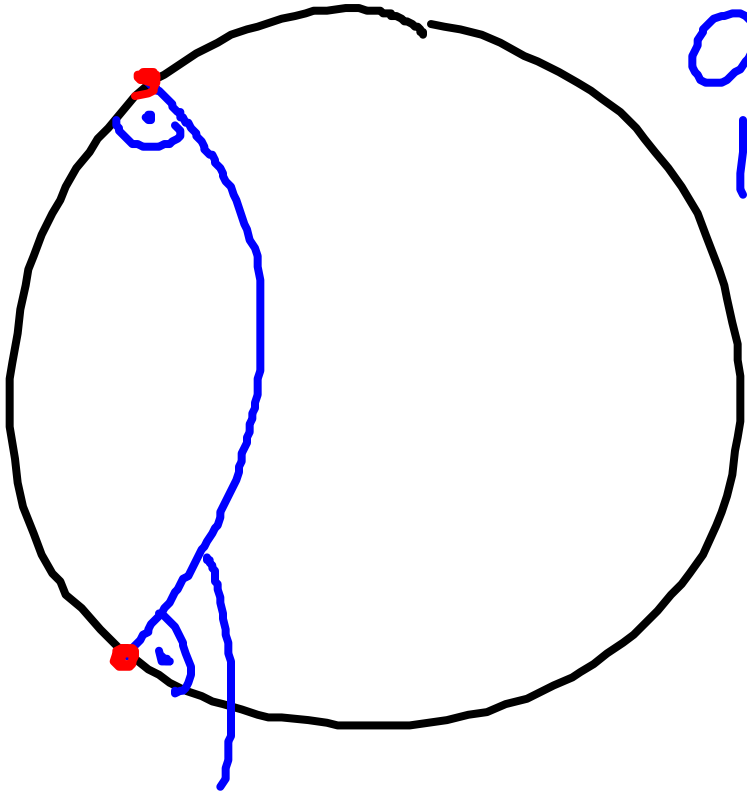
$x^2+y^2-z^2=0$

Secant to $\mathcal{F}$ are the lines

Hyp Transforms are those $\mathbb{RP}^2$ Transforms that leave $\mathcal{F}$ invariant.

## Poincaré Model

~ 1882



Circle C

$|z|=1$

in $\mathbb{CP}^1$

lines are circular arcs perpendicular to the boundary

Hyp Transforms are those $\mathbb{CP}^1$ Trafos that leave C invariant

How are these two models related?