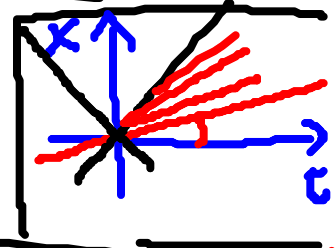
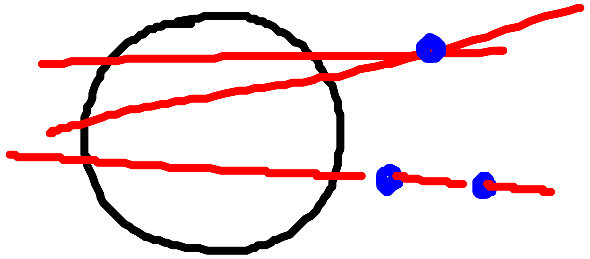
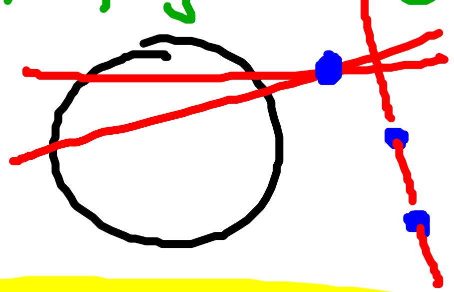
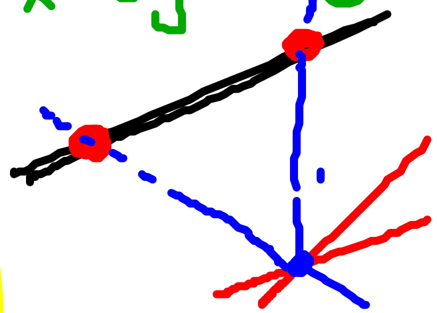
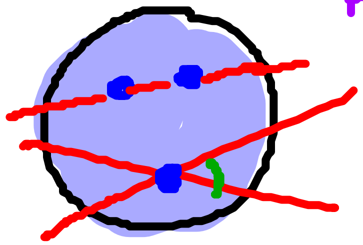
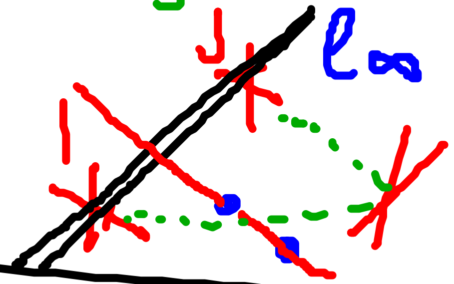
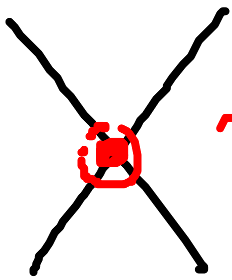
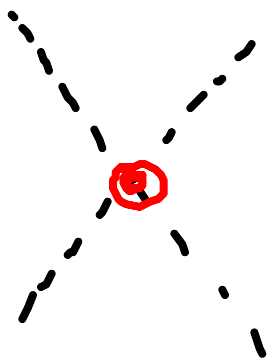
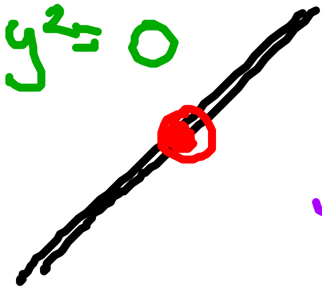
Classification of Cayley Klein geometries:



$\lvert P,Q \rvert$ \ $\measuredangle(L,M)$	Hyp	Ell	Euc
Hyp	$x^2+y^2-z^2=0$ $x^2+y^2-z^2=0$ 	$x^2+y^2-z^2=0$ $x^2+y^2-z^2=0$ 	$z^2=0$ $x^2-y^2=0$  Pseudo eucl. geometry ~ Special Relativity
Ell	$x^2+y^2-z^2=0$ $x^2+y^2-z^2=0$  Hyperbolic geometry	$x^2+y^2+z^2=0$ $x^2+y^2+z^2=0$ Elliptic geometry Geometry on a Sphere antipodes identified	$z^2=0$ $x^2+y^2=0$  ℓ_∞ Euclidean geometry
Euc	 Dual Pseudo Euklidean	 Dual Eukl.	$z^2=0$ $y^2=0$  Galileo geometry

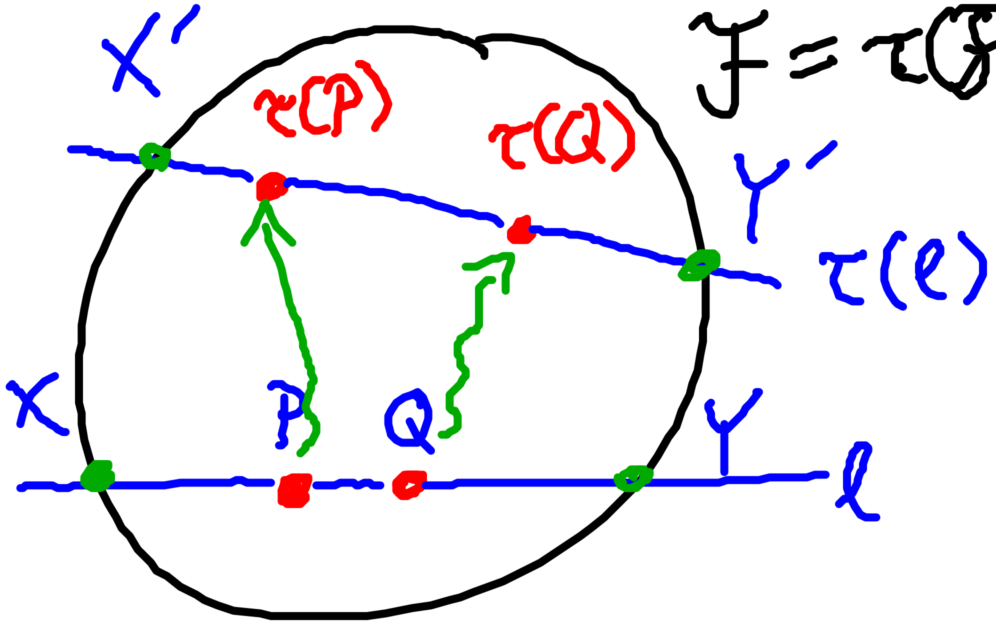
Transformations in CK geometries

$\mathcal{F} = (A, B)$ Fundamental Object.

Def Motions in a CK geometry are those projective Trafos that leave $\mathcal{F}$ as a whole invariant.

Thm: Under such Motions all distances and angles remain invariant (up to sign)

„Proof by example"



$\mathcal{F} = \tau(\mathcal{F})$

$cr(PQ; XY)$

$cr(\tau(P)\tau(Q), X', Y')$ =

Under τ we have:

$\{X', Y'\} = \{\tau(X), \tau(Y)\}$

Let $X' = \tau(X)$, $Y' = \tau(Y)$

=

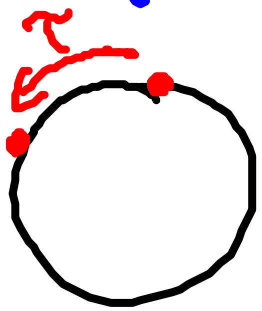
$cr(\tau(P), \tau(Q); \tau(X), \tau(Y))$

Since τ is a Proj. Trafo

Transformations of special CK-Geometries

1) Hyperbolic:

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$



Transformations that leave the unit circle as a whole invariant

This group of Trafos is isomorphic to $\mathbb{R}P^1$ Trafos

2 Elliptic Geometry

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad B : \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

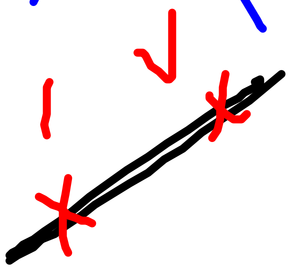
Transformations that leave $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ invariant

$$T^T E T = E \Rightarrow T^T T = E$$

These are Rotations in $\mathbb{R}^3$ of the homog. coord.

3 Euklidean

$$A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$



Transformations that leave I, J as a pair invariant.

$\Rightarrow$ Euclidean Trafos + Skaling + Mirror reflections

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4) Pseudo eukl. geom

$$A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

Trafos that leave $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ invariant

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5 + 6 dual to 3, 4

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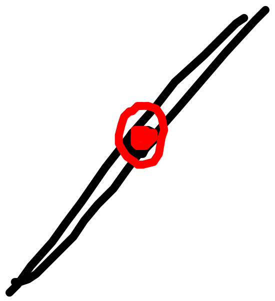
7 Gallileo geom



$$A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

Trafos that leave a line and a point on it fixed