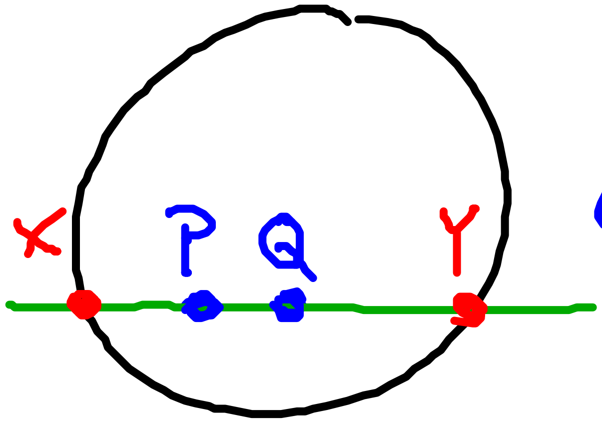
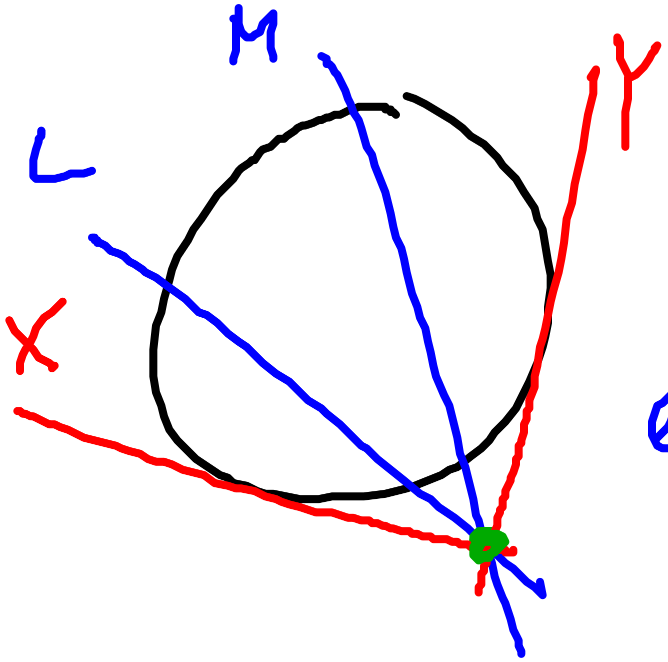
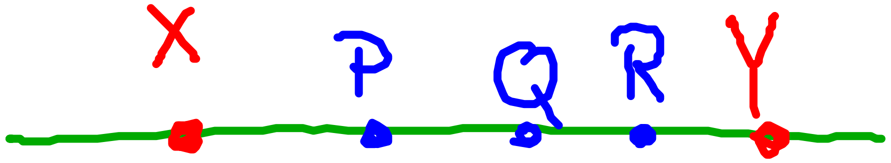
Cayley Klein Geometries: (A, B) Conic, cdist, cang



$|P,Q| :=$ cdist $\cdot \ln(PQ, XY)$



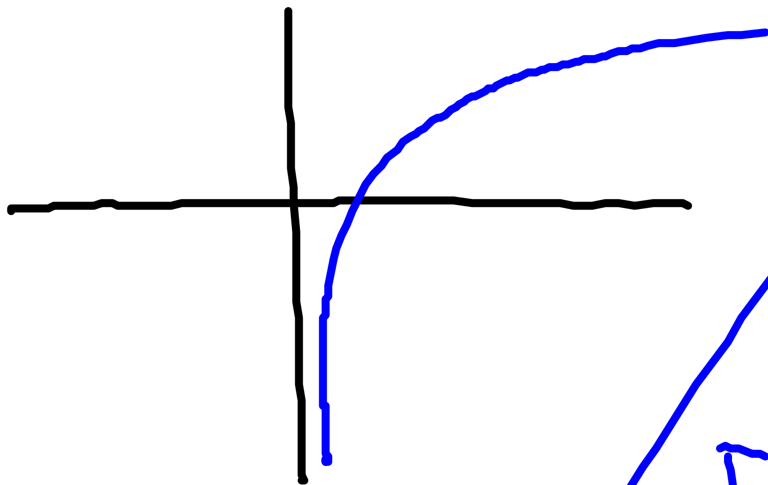
$\measuredangle L, M :=$ cang $\cdot \ln(LM, XY)$



- $|PQ|_{X,Y} = -|QP|_{XY}$
- $|PP|_{X,Y} = 0$
- $|PQ|_{X,Y} + |QR|_{X,Y} = |PR|_{XY}$

modulo cdist $\cdot 2\pi i$

$\ln \; \mathbb{R}^+ \to \mathbb{R}$



Calculus:

ln is the inverse Funktion to exp(x)

3 Types of measurement:

- X, Y both real $\Rightarrow$ Crossratio is Real: $\ln(CR)\begin{cases}\text{either real}\\ \text{or } r+\pi i\end{cases}$

  hyperbolic
- X, Y are complex conjugate $\Rightarrow$ crossratio is point on unit circle, $\ln(CR)$ is purely imag

  elliptic
- $X = Y$ are real $\Rightarrow$ crossratio $= 1$: $\ln(CR) = 0$

  euclidean

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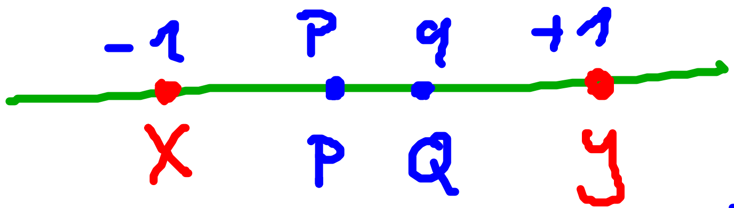
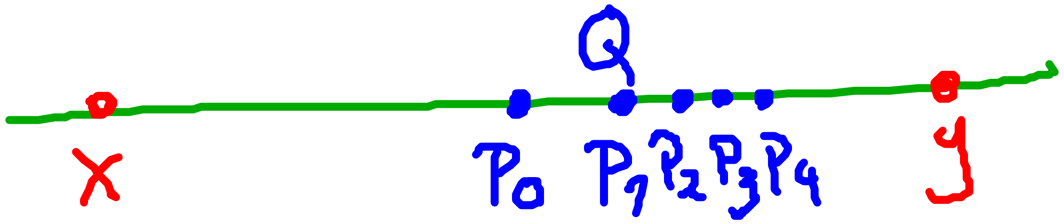
(1) Hyperbolic measurement.



$$|PQ| = c_{dist} \ln\left(\frac{(p+1)(q-1)}{(p-1)(q+1)}\right)$$

Special case $p = 0$

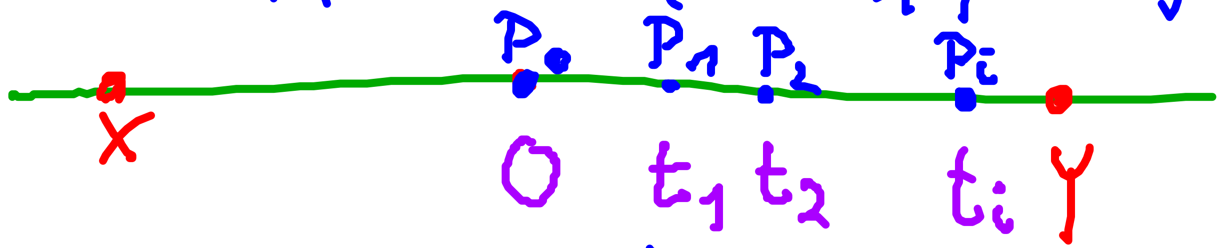
$$|PQ| = c_{dist} \ln\left(\frac{1-q}{1+q}\right)$$



$$|P_i P_{i+1}| = \text{const}$$

$$(P_0 P_1, X Y) = (P_i P_{i+1}, X Y)$$

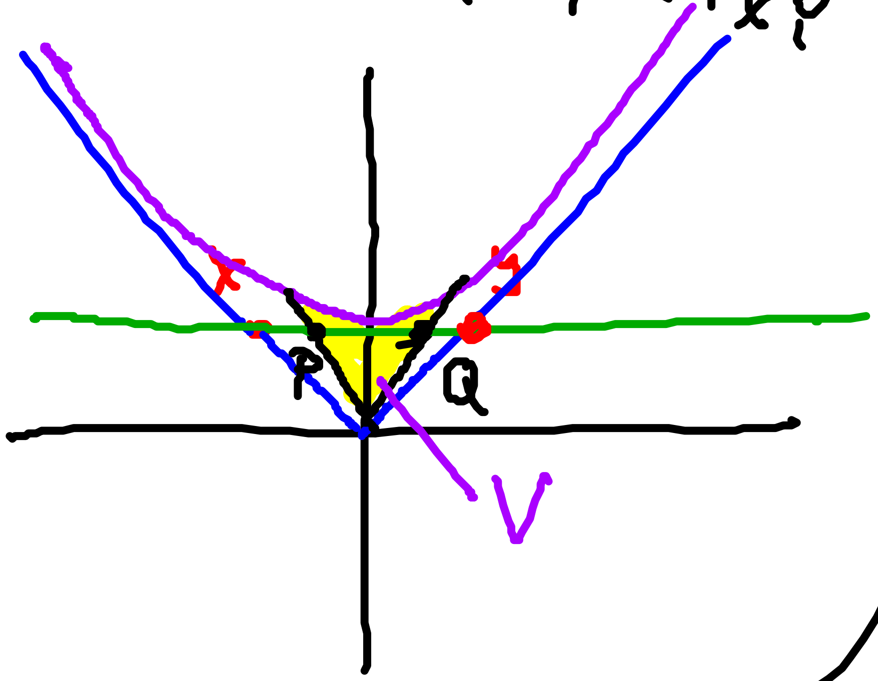
$$(P_0 P_i, X Y) = (P_0 P_1, X Y)^i$$



$$\left(\frac{1-t_i}{1+t_i}\right) = \Bigg(\underbrace{\frac{1-t_1}{1+t_1}}_{\alpha}\Bigg)^i \Rightarrow t_i = -\frac{\alpha^i - 1}{\alpha^i + 1}$$

Thm: Let $X = \begin{pmatrix}-1\\1\end{pmatrix}$, $Y = \begin{pmatrix}1\\1\end{pmatrix}$, $c_{dist} = -\frac{1}{2}$

then $|P,Q|_{XY} = 2 \cdot V$



Proof: wlog: $P = 0$

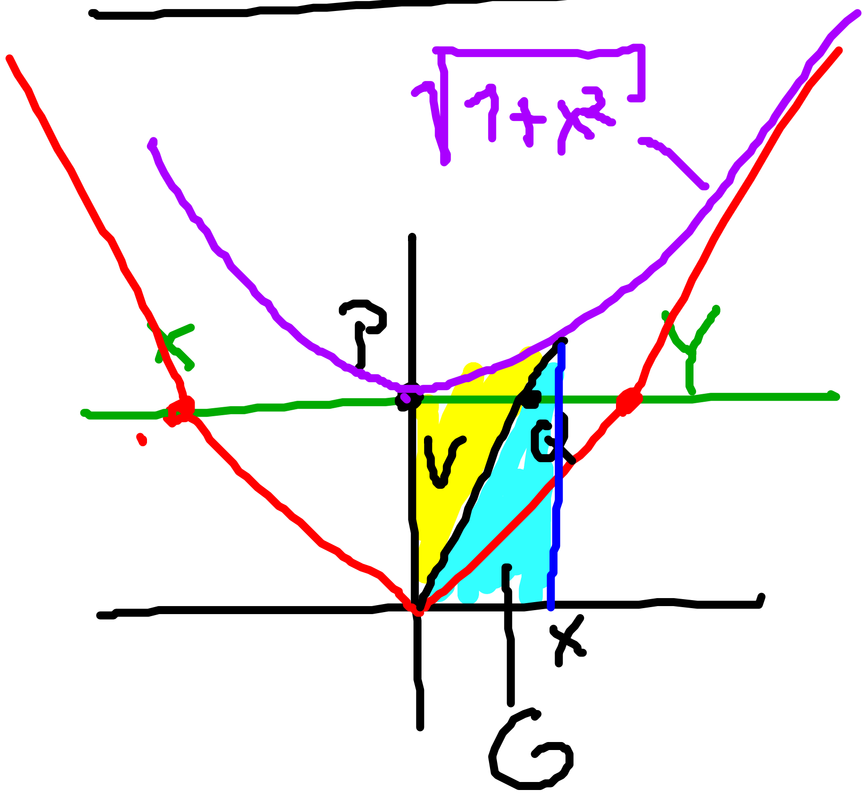
$$V = \int_0^x \sqrt{1+x^2}\,dx - G$$

$$= \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x+\sqrt{1+x^2}\right)\right) - x \cdot \sqrt{1+x^2} \cdot \frac{1}{2}$$

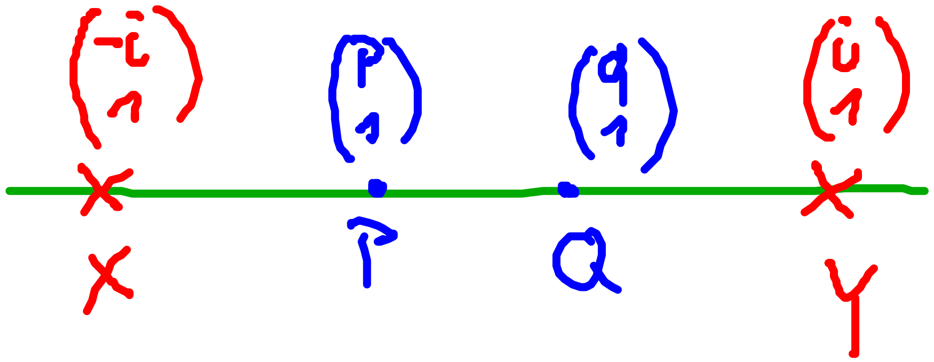
$$= \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)$$

⋮ a miracle occurs

$$= |P,Q|_{XY}$$



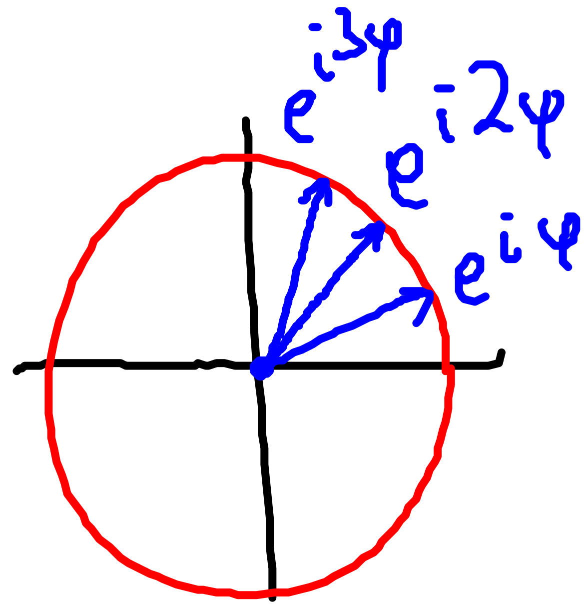
Elliptic Measurement:



complex conjugates

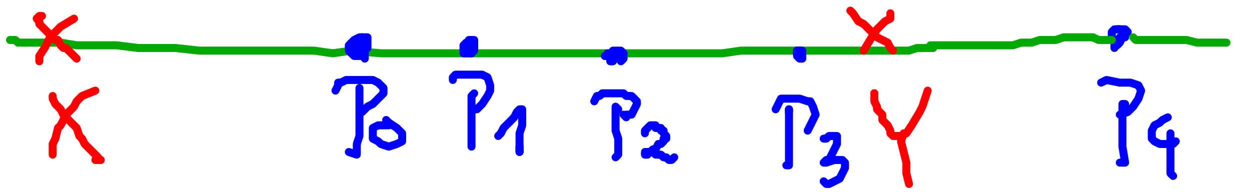
$$|P,Q|_{X,Y} = c_{dist} \ln\left(\frac{(p+i)(q-i)}{(p-i)(q+i)}\right) = e^{i\varphi} \qquad \varphi \in \mathbb{R}$$

„good" c_{dist} is $\frac{1}{2i}$



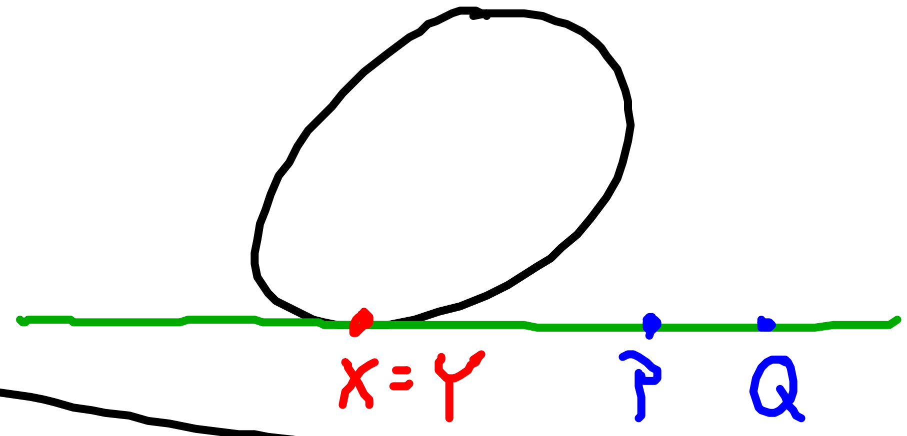
$$(P_0 P_1, XY) = \qquad = e^{i\varphi}$$

$$(P_0 P_j, XY) = \dots = (e^{i\varphi})^j$$



What happens if $X = Y$?

$\mathrm{cdist}\ \ln(P,Q;XX) = \mathrm{cdist}\ \ln(1) = 0$



Way out to get at least some relative measurement: compare to „a unit length“

$X = \begin{pmatrix}1\\ -\sqrt{\alpha}\end{pmatrix}$ $\quad Y = \begin{pmatrix}1\\ +\sqrt{\alpha}\end{pmatrix}$ calculate $\lim \alpha \to 0$

$$\ln(\underset{\binom{0}{1}}{P},\underset{\binom{q}{1}}{Q};XY) = \ln\left(\frac{q\sqrt{\alpha}-1}{-q\sqrt{\alpha}-1}\right) = \ln(q\sqrt{\alpha}-1) - \ln(-q\sqrt{\alpha}-1)$$

X, Y real

hyp. Meas.

$\begin{pmatrix}1\\1\end{pmatrix} = X, \quad \begin{pmatrix}-1\\1\end{pmatrix}\ Y$

X, Y complex conj

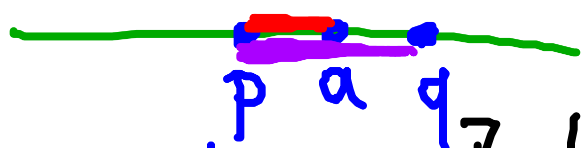
ell. Meas.

$\begin{pmatrix}-i\\1\end{pmatrix} = X \quad \begin{pmatrix}i\\1\end{pmatrix}\ Y$

Solutions of

$\alpha x^2 - y^2 = 0$

Comparison of lengths:



$$\lim_{\alpha\to 0}\frac{\ln(P,q,XY)}{\ln(P,a,XY)} = \underset{\text{l'Hospital}}{\cdots\cdots} = \frac{q}{a}$$

$\Leftarrow$ Behaves as in Euclidean geometry