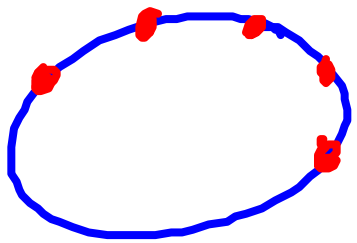
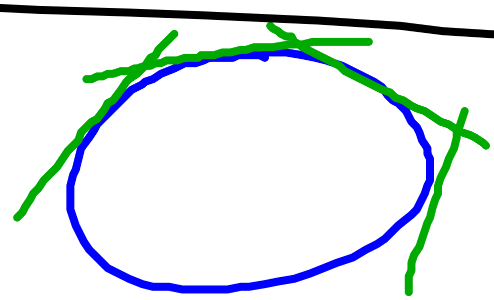
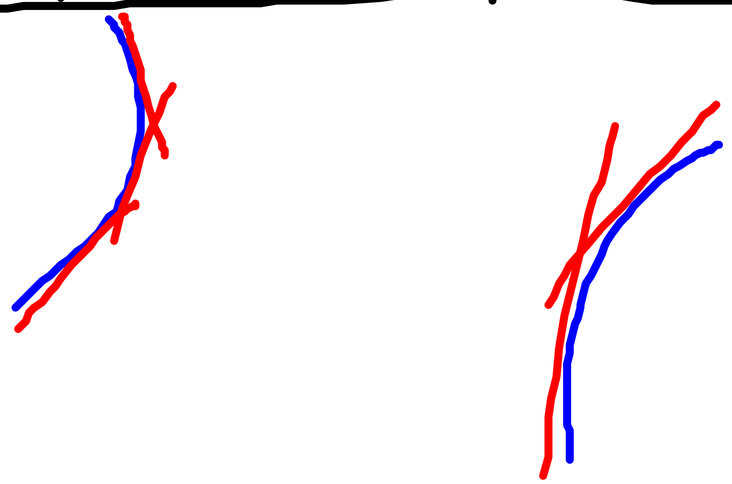
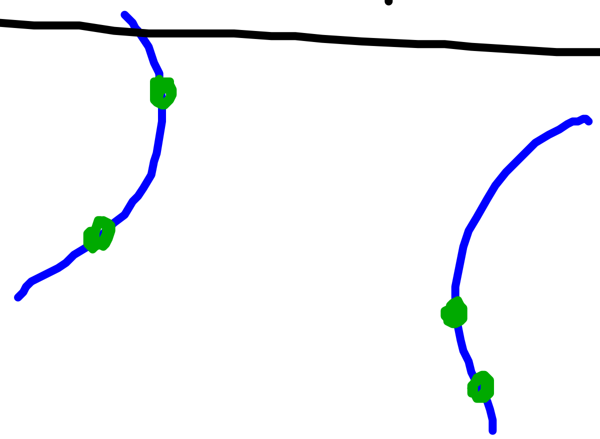
Dual Conic $B = A^{-1}$ (non-degenerate)

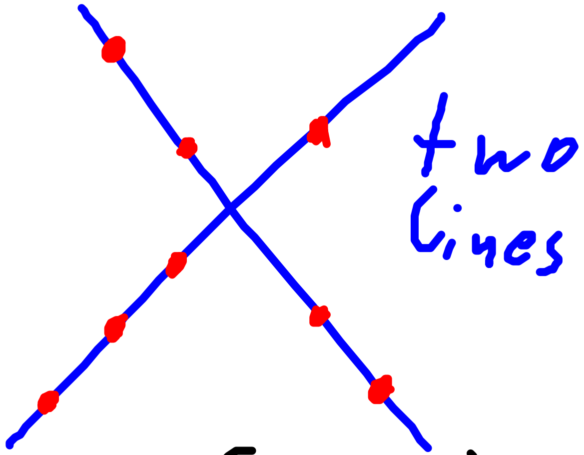
A different matrix which describes the same conic in terms of its incident lines (i.e. its tangents) instead of its incident points.

	primal conic	dual conic
primal view	 $\{p \in \mathcal{P} \mid p^T A p = 0\}$	 $\{\ell \in \mathcal{L} \mid \ell^T B \ell = 0\}$
dual view		

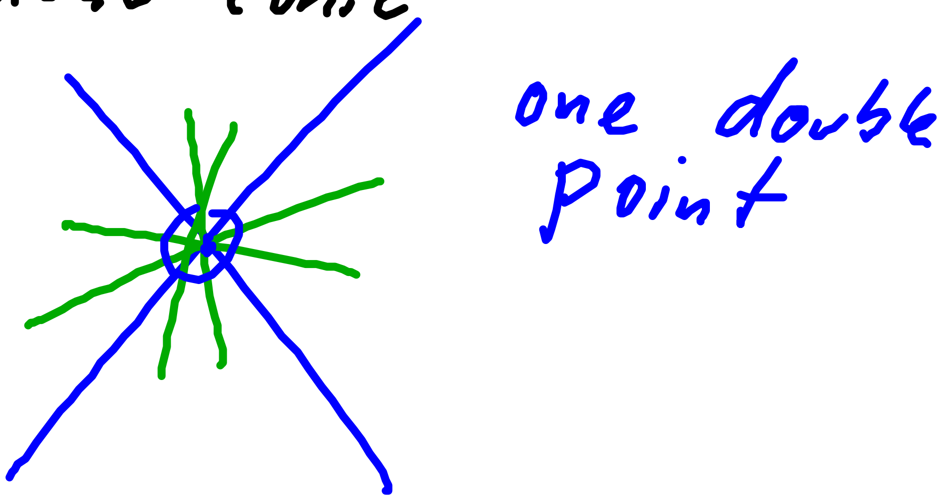
Degenerate example

primal conic → one way → dual conic

primal view

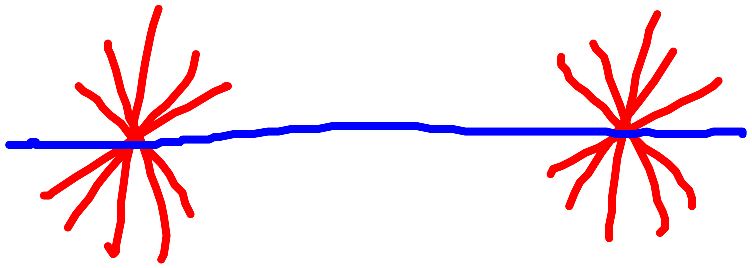


$A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ $x^2 = y^2$



$B = A^\Delta = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -1 \end{pmatrix}$ $z^2 = 0$

dual view



two points

a double line

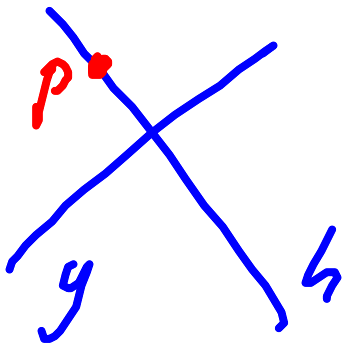
Dual conic in degenerate cases:

The adjugate (or classical adjoint) matrix is

$$A^{\Delta} := \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

It is a multiple of A^{-1} in the non-degenerate cases but is still defined in degenerate cases and non-zero if $\operatorname{rank}(A) \geq 2$

Pair of lines → conic



$$0 = \langle p, g \rangle \langle h, p \rangle = p^T \cdot (g \cdot h^T) \cdot p$$

$g \cdot h^T$ rank 1 unsymmetric

$A = g \cdot h^T + h \cdot g^T$ rank 2 symmetric

Conic → pair of lines

$$A + \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix}$$

describes the same conic in an unsymm. way.

Find λ, μ, τ such that its rank is 1

To check: every 2×2 determinant is zero

Then: g is any non-zero column and
h is any non-zero row of that matrix

$$\operatorname{rank}\begin{pmatrix} a & d+\tau & e-\mu \\ d-\tau & b & f+\lambda \\ e+\mu & f-\lambda & c \end{pmatrix} = 1 \quad \Rightarrow \quad \begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix} = \begin{pmatrix} \pm\sqrt{-\begin{vmatrix} b & f \\ f & c \end{vmatrix}} \\ \pm\sqrt{-\begin{vmatrix} a & e \\ e & c \end{vmatrix}} \\ \pm\sqrt{-\begin{vmatrix} a & d \\ d & b \end{vmatrix}} \end{pmatrix}$$

$$\begin{vmatrix} b & f+\lambda \\ f-\lambda & c \end{vmatrix} = 0$$

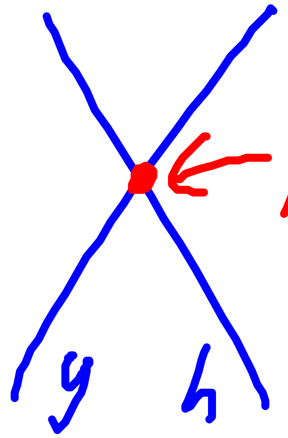
$$b\cdot c - f^2 + \lambda^2 = 0$$

$$\lambda^2 = -\begin{vmatrix} b & f \\ f & c \end{vmatrix}$$

$$\lambda = \pm\sqrt{-\begin{vmatrix} b & f \\ f & c \end{vmatrix}}$$

How to choose signs consistently?

„Surprising fact"



$$p = \begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix} = \alpha\,(g \times h)$$

If one knows $p = g \times h = \begin{pmatrix} \lambda' \\ \mu' \\ \tau' \end{pmatrix}$ then one can obtain $\begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix}$ as follows:

set $\alpha = \dfrac{\sqrt{-\begin{vmatrix} b & f \\ f & c \end{vmatrix}}}{\lambda'}$ (w.l.o.g. $\lambda' \neq 0$)

(First sign is arbitrary)

then compute

$$\mu = \alpha \cdot \mu'$$

$$\tau = \alpha \cdot \tau'$$

Proof of this „surprising fact":

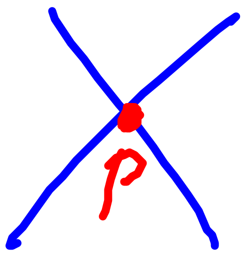
Let $A = g\cdot h^T + h\cdot g^T$, $\quad p = g\times h = \begin{pmatrix} g_2h_3 - g_3h_2 \\ g_3h_1 - g_1h_3 \\ g_1h_2 - g_2h_1 \end{pmatrix} \overset{\alpha=1}{=} \begin{pmatrix} \lambda \\ \mu \\ \tau \end{pmatrix}$

Conside

$$g\cdot h^T - h\cdot g^T = \begin{pmatrix} 0 & g_1h_2-g_2h_1 & g_1h_3-g_3h_1 \\ g_2h_1-g_1h_2 & 0 & g_2h_3-g_3h_2 \\ g_3h_1-g_1h_3 & g_3h_2-g_2h_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix} = M_p$$

Now $A + M_p = gh^T + hg^T + gh^T - hg^T = 2\cdot g\cdot h^T$ has rank 1

How to find p?



Every tangent passes through p.

$\Longrightarrow \quad A^\Delta = p\cdot p^T$

so compute A^Δ and read p as any non-zero row or column.

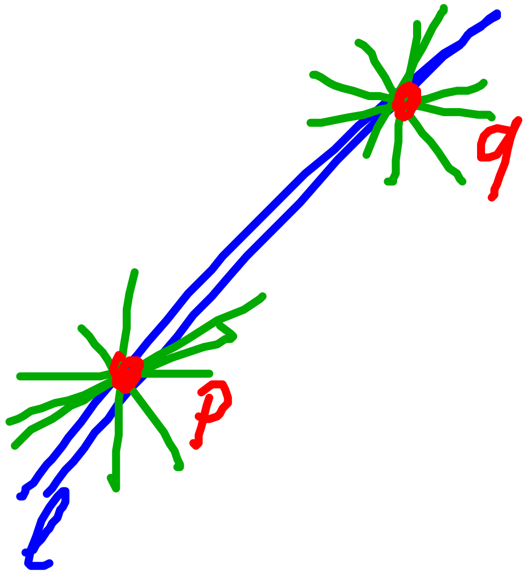
Cross product matrix

For $p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the matrix

$M_p = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$ satisfies

$$M_p\cdot q = p\times q$$

Dual situation:



Double line with two distinguished points encoded in B

All tangents: $\{\ell \in \mathcal{L} \mid \ell^T \cdot B \cdot \ell\}$

matrix of dual conic

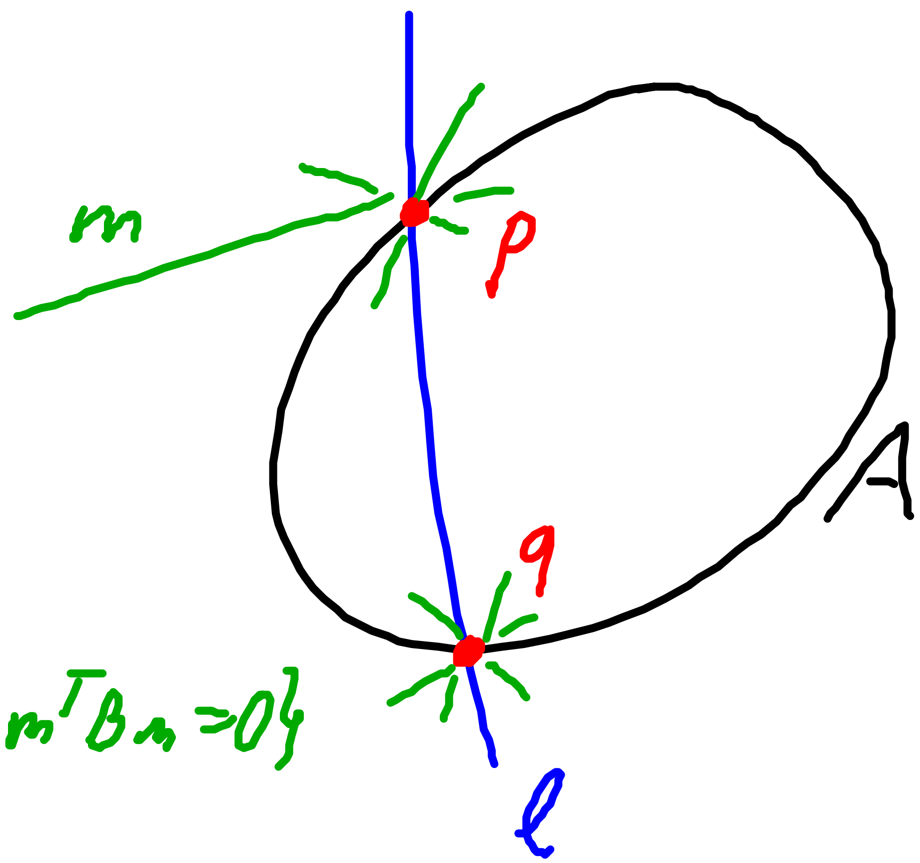
Decomposing this into a pair of points

1. Find $B' = B + \alpha \cdot M_\ell$ such that $\mathrm{rank}(B') = 1$
   (using $B^\Delta = \ell \cdot \ell^T$ if you don't know ℓ)
2. Decompose $B' = p \cdot q^T$ by reading a non-zero row & column

- p on A and on l
- $p^T \cdot A \cdot p = 0$
- $(l \times m)^T \cdot A \cdot (l \times m)$
- $(M_l \cdot m)^T \cdot A \cdot (M_l \cdot m)$
- $m^T \cdot (M_l^T \cdot A \cdot M_l) \cdot m$
- $B = M_l \cdot A \cdot M_l$

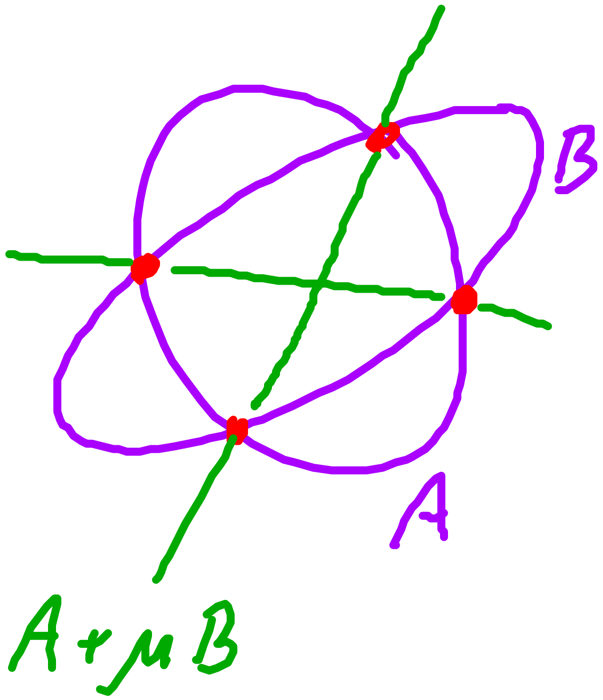
The set of lines passing through p or q is $\{m \mid m^T B m = 0\}$

- $B' = B + \alpha \cdot M_l$ such that $\mathrm{rank}(B') = 1$
- Choose any non-zero row and column of B' as p and q.



B is the matrix of a degenerate dual conic consisting of the two points of intersection.

Intersecting one conic with another conic



- Every conic of the form $\lambda \cdot A + \mu \cdot B$ will pass through the points of intersection.

$$\underbrace{p^T(\lambda A + \mu B)p}_{=0} = \lambda \cdot \underbrace{(p^T A p)}_{=0} + \mu \cdot \underbrace{(p^T B p)}_{=0}$$

$\Leftarrow$ $=0$ and $=0$

- Find λ, μ such that $\det(\lambda A + \mu B) = 0$

  If B is non-degenerate, w.l.o.g. $\lambda = 1$

  Then solve $\det(A + \mu B) = 0$ which is a 3rd degree equation in μ.

- Decompose $A + \mu B$ into two lines

- Either intersect these with one of the conics
  - Or choose another solution for μ and intersect one line from each pair