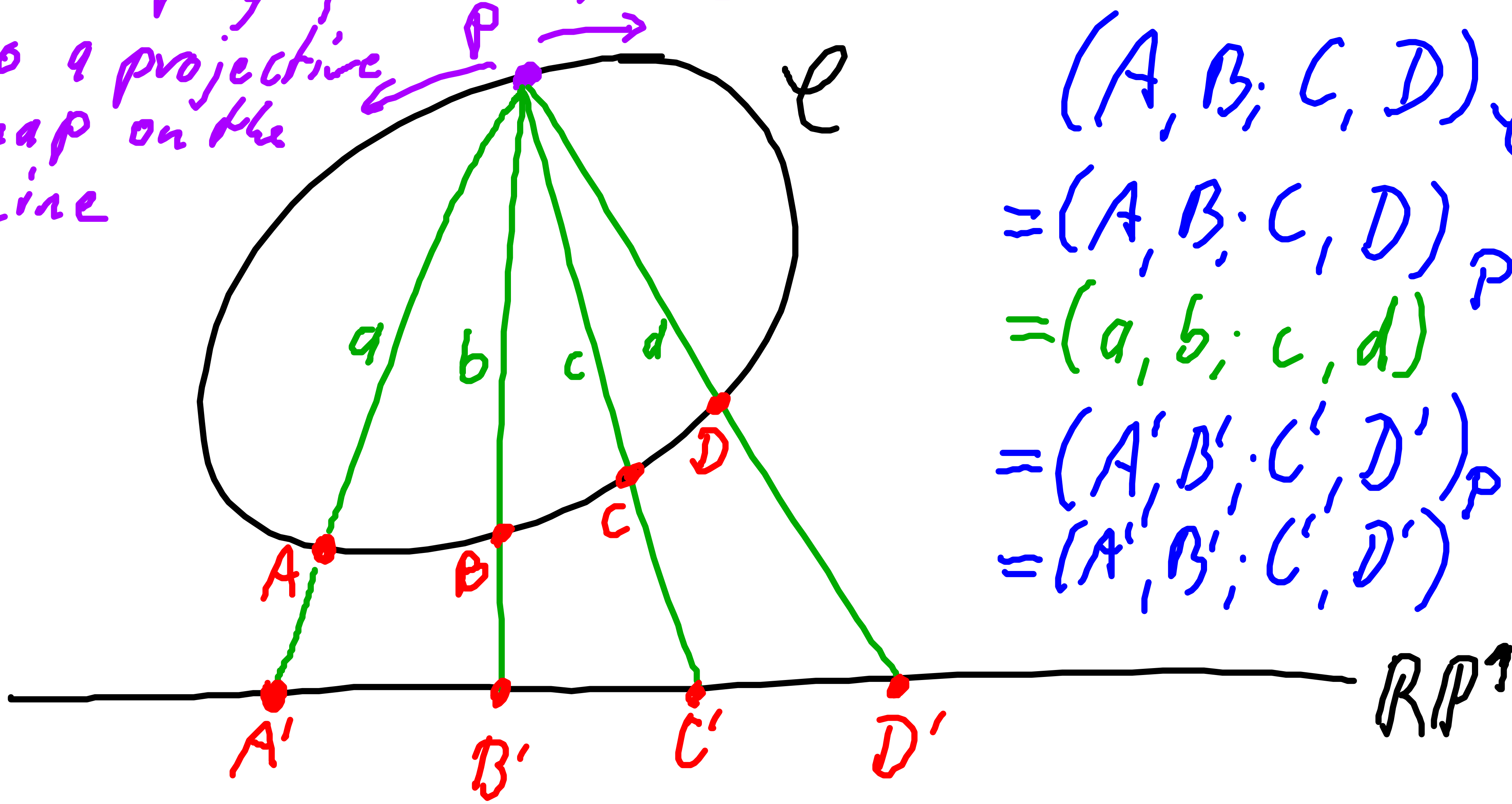
Changing P corresponds to a projective map on the line



$$(A,B;C,D)_{\mathscr{C}}$$
$$=(A,B;C,D)_P$$
$$=(a,b;c,d)$$
$$=(A',B';C',D')_P$$
$$=(A',B';C',D')$$

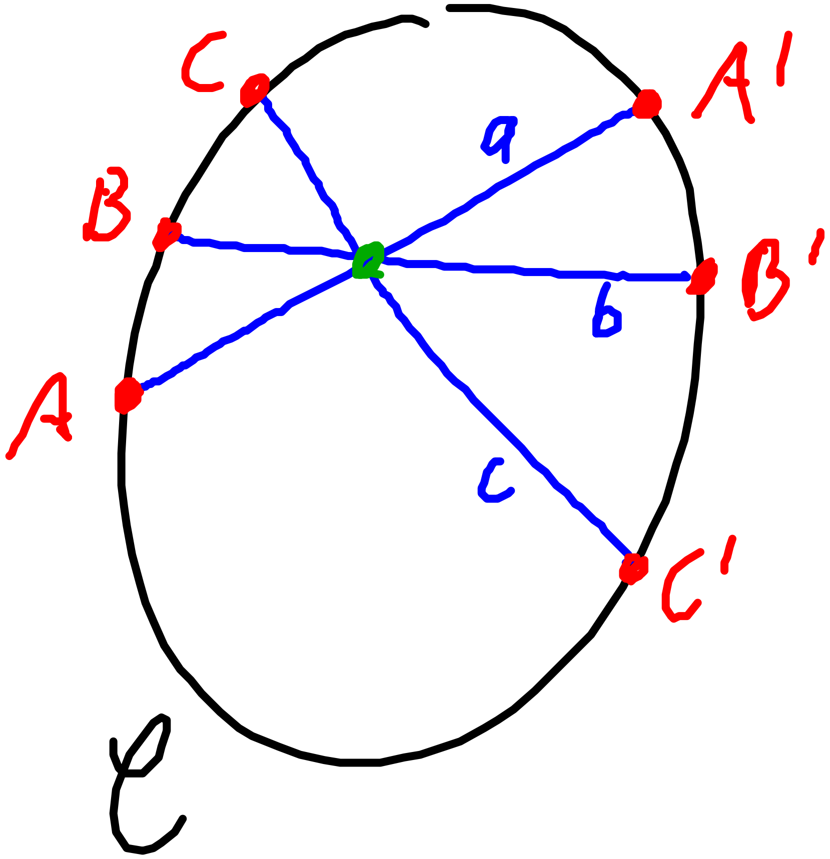
One-to-one map which preserves cross ratios

$\Rightarrow$ $\{$Points on $\mathscr{C}\}$ is isomorphic to $\mathbb{R}P^1$

Warnings:

- Only works for non-deg $\mathscr{C}$
- Map from $\mathscr{C}$ to line is <u>not</u> a proj. transf. of $\mathbb{R}P^2$

(One aspect of) Hesse's Übertragungsprinzip (transfer principle)



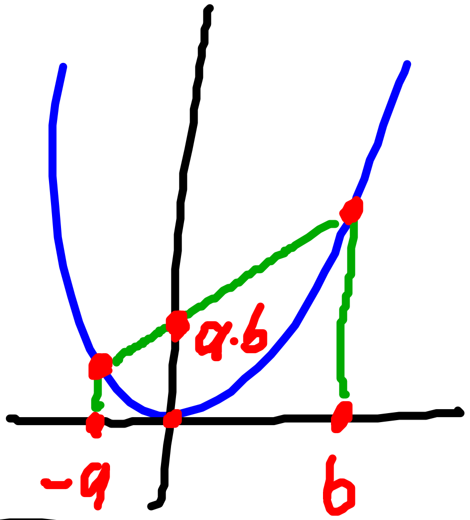
Thm: Let $\mathcal{C}$ be a real and non-degenerate conic and a, b, c three lines intersecting $\mathcal{C}$. Then a, b, c intersect in a single point if and only if the six points of intersection form a quadrilateral set

$$(A, A'; B, B'; C, C')$$

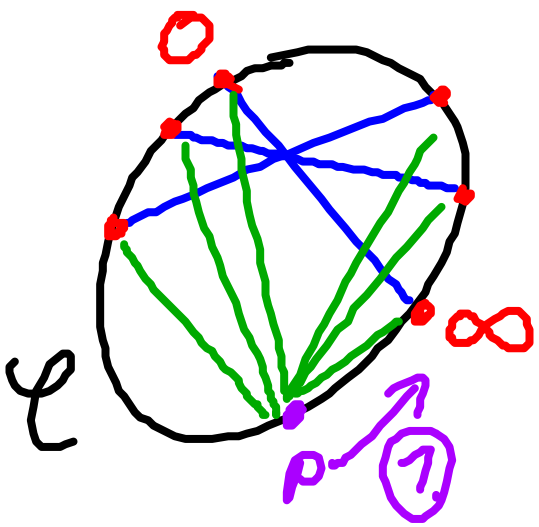
Proof by reduction to special case

$yz = x^2$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

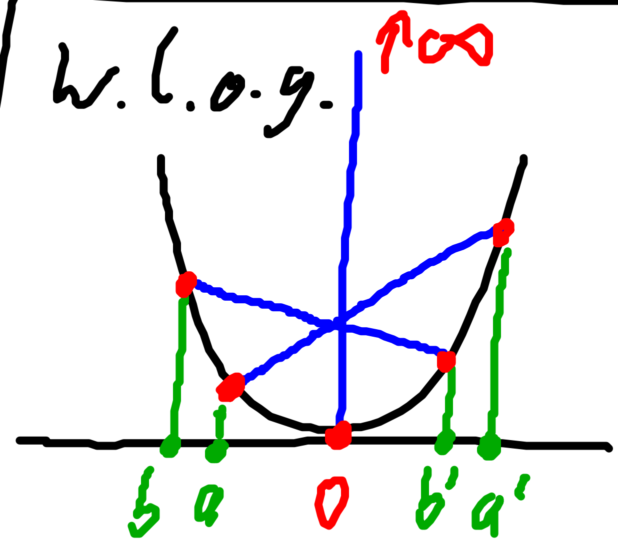


$$\begin{vmatrix} -a & b & 0 \\ a^2 & b^2 & a\cdot b \\ 1 & 1 & 1 \end{vmatrix} = 0$$



1. Move p to one of the six points
2. Move that point to ∞, its opposite point to zero and $\mathcal{C}$ to the parabola $yz = x^2$

w.l.o.g.



$(a, a'; b, b'; 0, \infty)$ are quad set if and only if

$$a \cdot a' = b \cdot b'$$

$$[a\, b'][b\, \infty][0\, a'] = [a\, \infty][b\, a'][0\, b']$$

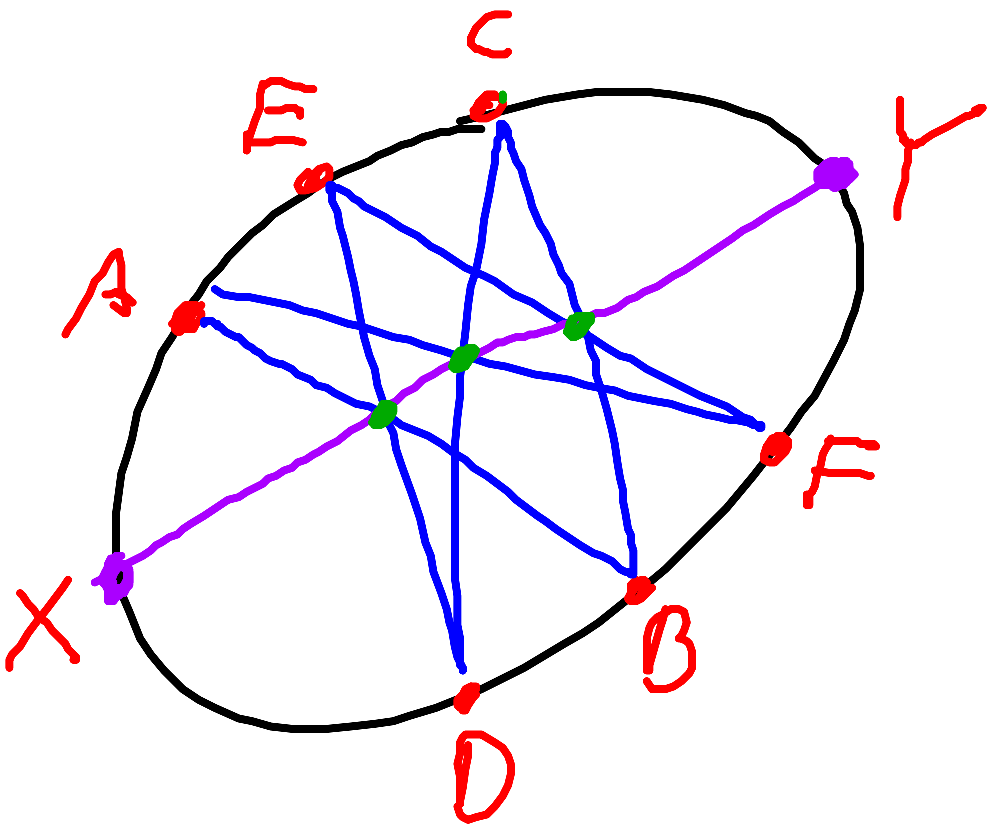
$$\begin{vmatrix} a & b' \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} b & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & a' \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} b & a' \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & b' \\ 1 & 1 \end{vmatrix}$$

$$(a - b') \cdot (-1) \cdot (-a') = (-1) \cdot (b - a') \cdot (-b')$$

$$aa' \cancel{- a'b'} = bb' \cancel{- a'b'}$$

$$a \cdot a' = b \cdot b'$$

Pascal's Theorem



Proving this thm using Hesse and quad sets

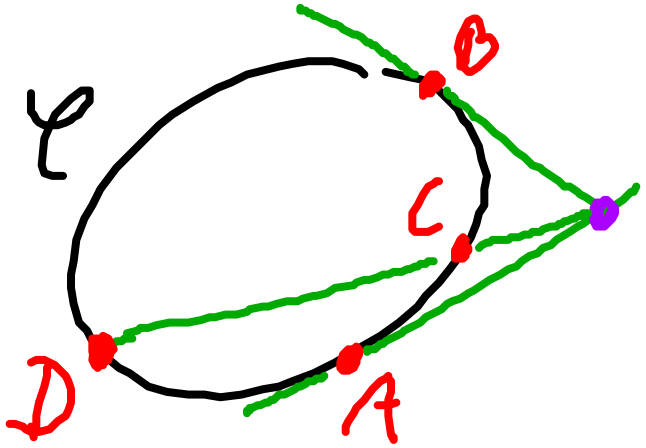
$$(A,B;E,D;X,Y) \Rightarrow \cancel{[AD]}[EY][XB] = \cancel{[AY]}[EB]\cancel{[XD]}$$

$$(C,D;A,F;X,Y) \Rightarrow [CF]\cancel{[AY]}\cancel{[XD]} = [CY]\cancel{[AD]}[XF]$$

$$(E,F;C,B;X,Y) \Leftarrow [EY][CF][XB] = [EB][CY][XF]$$

More applications:

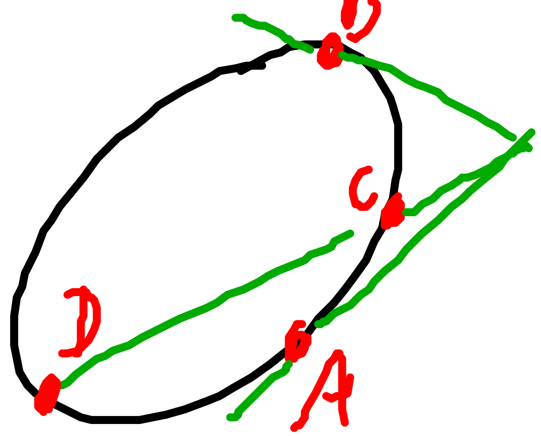
- Addition, multiplication à la von Staudt
- Construction of a harmonic point

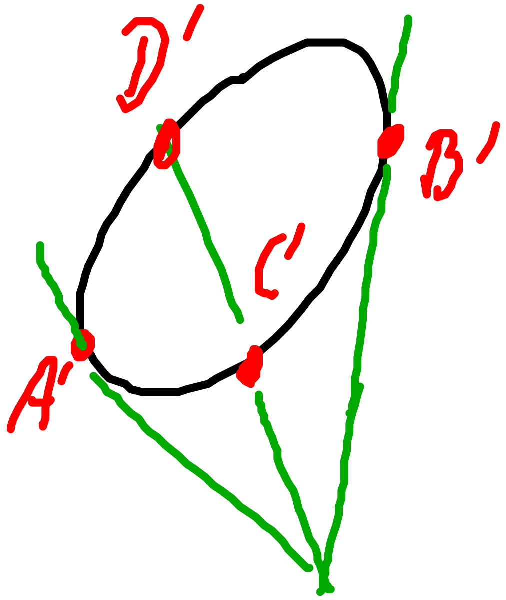


$(A,A;\ B,B;\ C,D)$ are quad set

$(A,B;C,D) = -1$

- Finding proj. transformations which fix $\mathcal{C}$





Theorem: Let $\mathcal{C}$ be real and non-degenerate conic and τ be a projective transformation of $\mathbb{RP}^2$ which fixes $\mathcal{C}$

$\tau : \mathbb{RP}^2 \to \mathbb{RP}^2$ $\quad \tau(\mathcal{C}) = \mathcal{C}$ $\quad (\mathcal{C}: p^T A p = 0)$

$\tau : p \mapsto Mp$ $\quad (M^{-1})^T A (M^{-1}) = \lambda A$

Then τ acts on $\mathcal{C}$ as a proj. transf. on $\mathbb{RP}^1$

Proof: $(A,B;C,D)_{\mathcal{C}} \overset{\text{for any point } p \text{ on } \mathcal{C}}{=} (A,B;C,D)_p = (\tau(A),\tau(B);\tau(C),\tau(D))_{\tau(p)}$

because τ is proj. transf. in $\mathbb{RP}^2$

$= (\tau(A),\tau(B);\tau(C),\tau(D))_{\mathcal{C}}$

since $\tau(p) \in \mathcal{C}$ if $p \in \mathcal{C}$

$\Rightarrow$ Cross ratio is preserved

$\Rightarrow$ projective transformation on $\mathbb{RP}^1$ $\square$

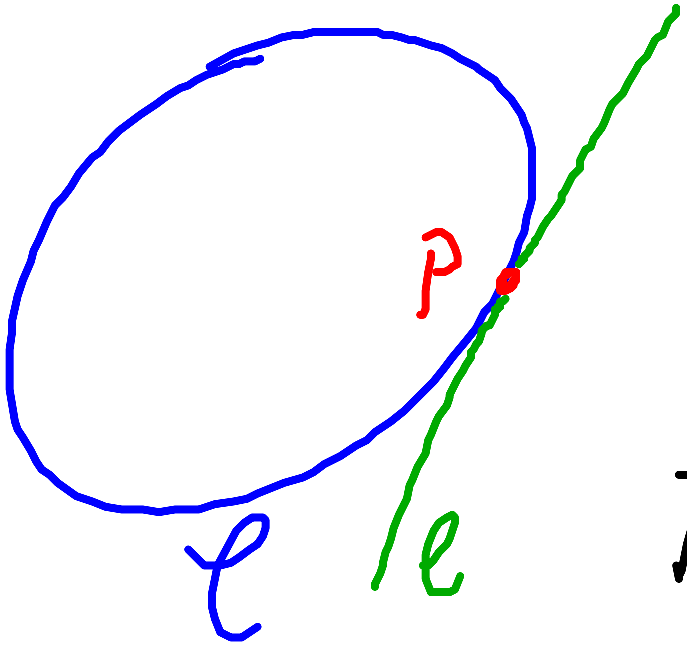
Thm: Let (A,B,C,D) and (A',B',C',D') be two quadruples of points on a real and non-degenerate conic $\mathcal{C}$ which satisfy $(A,B;C,D)_{\mathcal{C}} = (A',B';C',D')_{\mathcal{C}}$.
Then the uniquely determined projective transform. τ with $\tau(A)=A', \dots, \tau(D)=D'$ also satisfies $\tau(\mathcal{C}) = \mathcal{C}$.

Proof: Uniquely determined $\tau \longrightarrow$ Geometriekalküle

$$\forall p \in \mathcal{C} : \tau(p) \in \mathcal{C}$$

$$(A,B;C,D)_p = \begin{cases} (\tau(A),\tau(B);\tau(C),\tau(D))_{\tau(p)} = (A',B';C',D')_{\tau(p)} \\ (A,B;C,D)_{\mathcal{C}} \underset{\substack{\uparrow \\ \text{by assumption}}}{=} (A',B';C',D')_{\mathcal{C}} \end{cases} \Rightarrow \tau(p) \in \mathcal{C}$$

Polarity and Duality



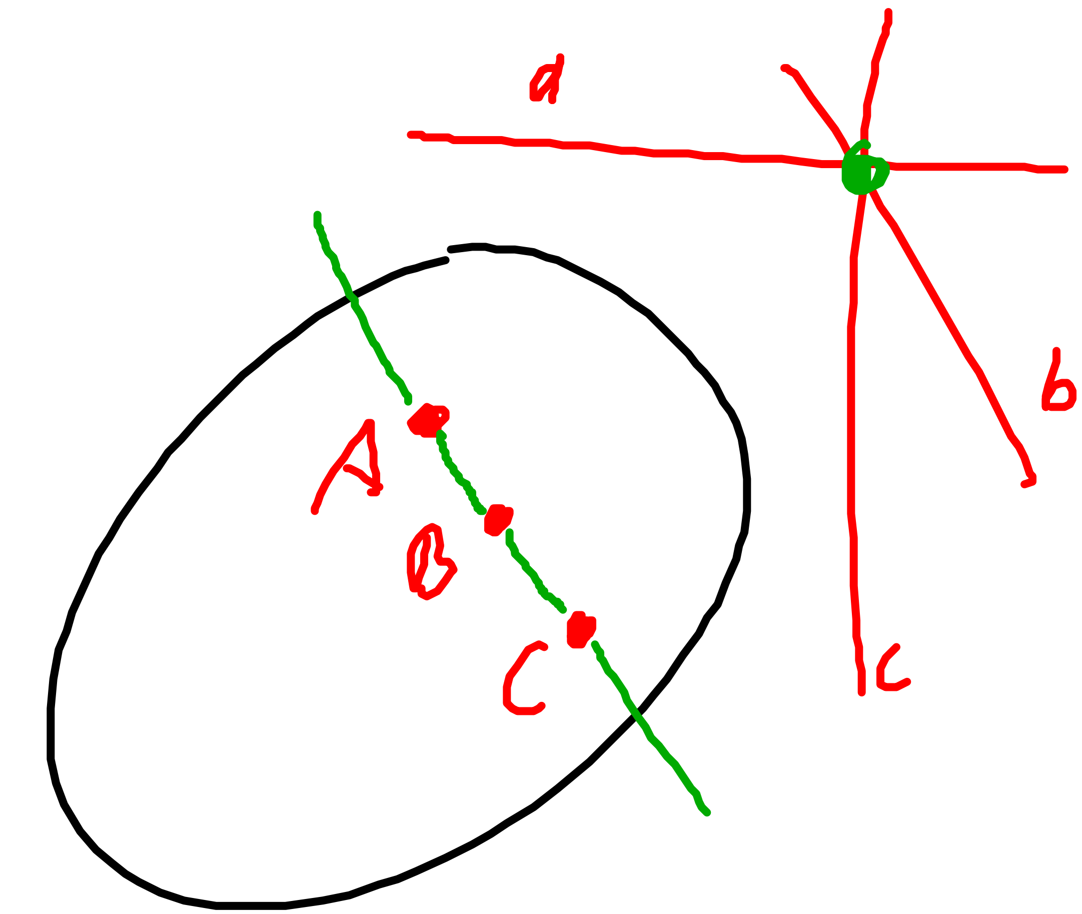
$\mathcal{C} : \{ p \in \mathcal{P} \mid p^T \cdot A \cdot p = 0 \}$

for some symmetric matrix A, $\det(A) \neq 0$

The tangent ℓ to $\mathcal{C}$ in p is $A \cdot p$ interpreted as homogeneous coordinates of a line

If p is <u>not</u> on $\mathcal{C}$, then the line $\ell = A \cdot p$ is called the polar of p (with respect to $\mathcal{C}$). Conversely, $p = A^{-1} \cdot \ell$ is called the pole of ℓ.

The map $\mathcal{P} \to \mathcal{L}$, $p \mapsto A \cdot p = \ell$ is called polarity



$$p^T M p = 0$$

$$\det(M) \neq 0$$

A, B, C collinear

$$\iff [A, B, C] = 0$$

$$\iff \det(M) \cdot [A, B, C]$$

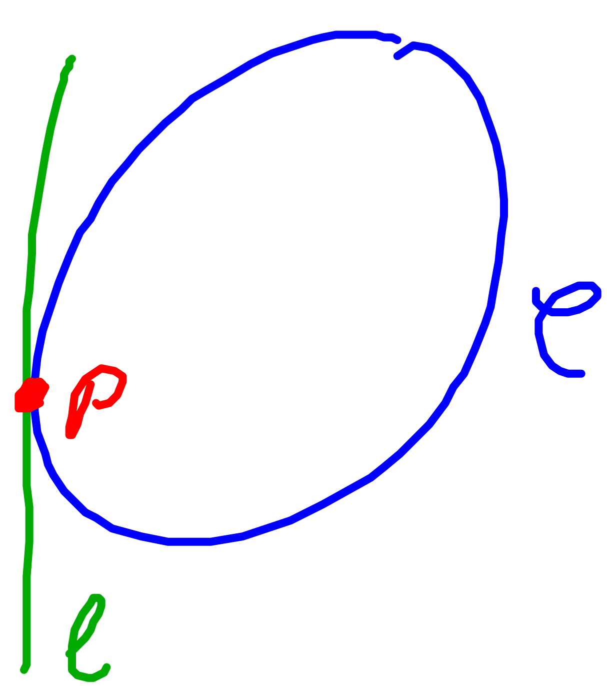
$$\iff [M \cdot A, M \cdot B, M \cdot C] = 0$$

$$\iff [a, b, c] = 0$$

$\iff a, b, c$ are concurrent

The set of all tangents (non-degenerate case)

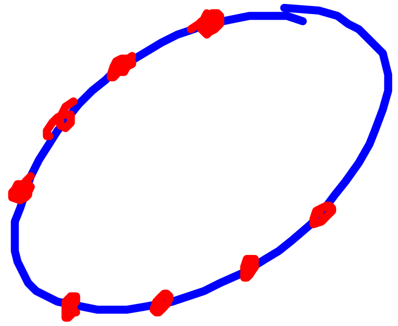
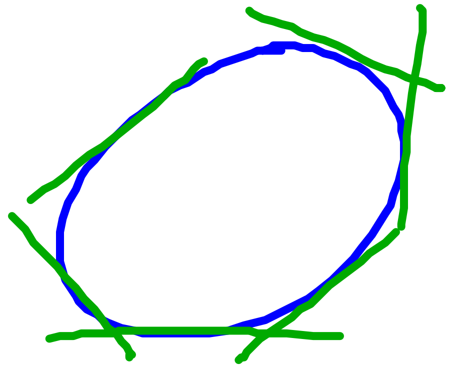
ℓ is a tangent to $\mathcal{C}$ if and only if there exists p on $\mathcal{C}$ such that ℓ is its polar line.



$$\ell = A \cdot p \qquad p^T \cdot A \cdot p = 0$$

$$p = A^{-1} \cdot \ell \qquad \ell^T \cdot A^{-1} \cdot A \cdot A^{-1} \cdot \ell = 0$$

$\{\ell \in \mathcal{L} \mid \ell^T \cdot A^{-1} \cdot \ell = 0\}$ is the set of tangents

	primal conic	dual conic
primal view		
	$\{p \in \mathcal{P} \mid p^T \cdot A \cdot p = 0\}$	$\{\ell \in \mathcal{L} \mid \ell^T B \ell = 0\}$ $B = A^{-1}$
dual view