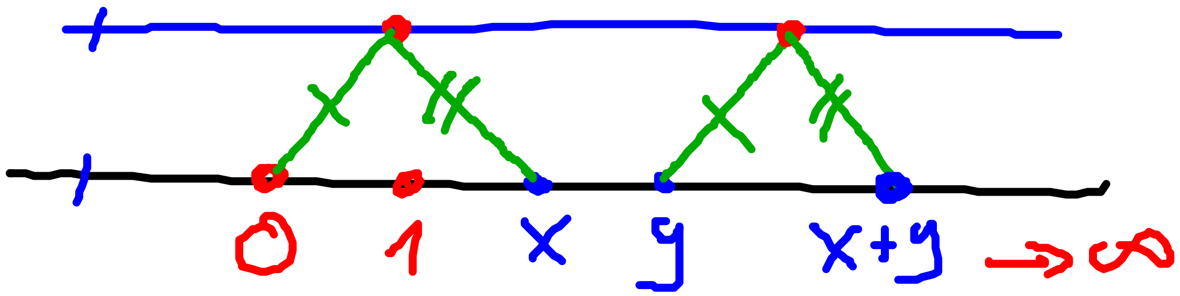
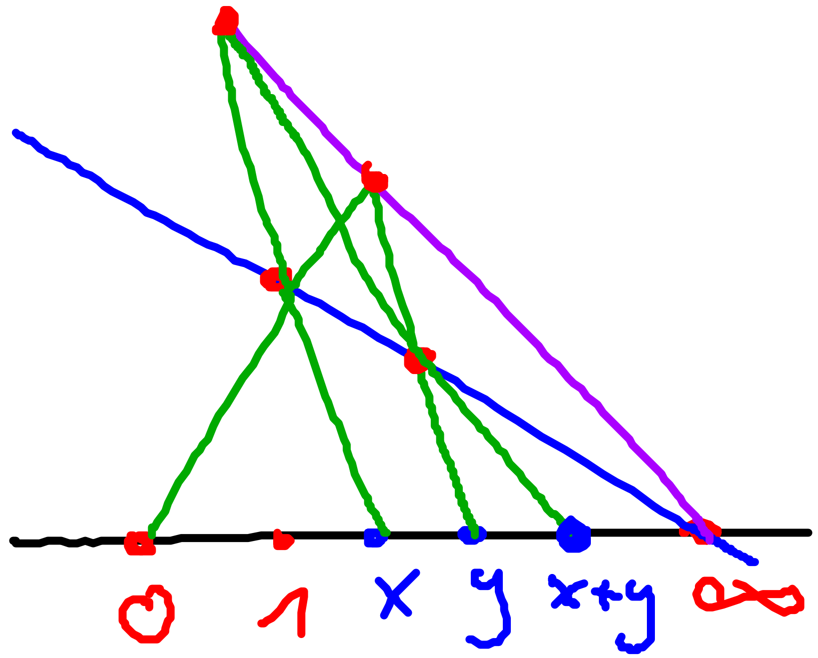
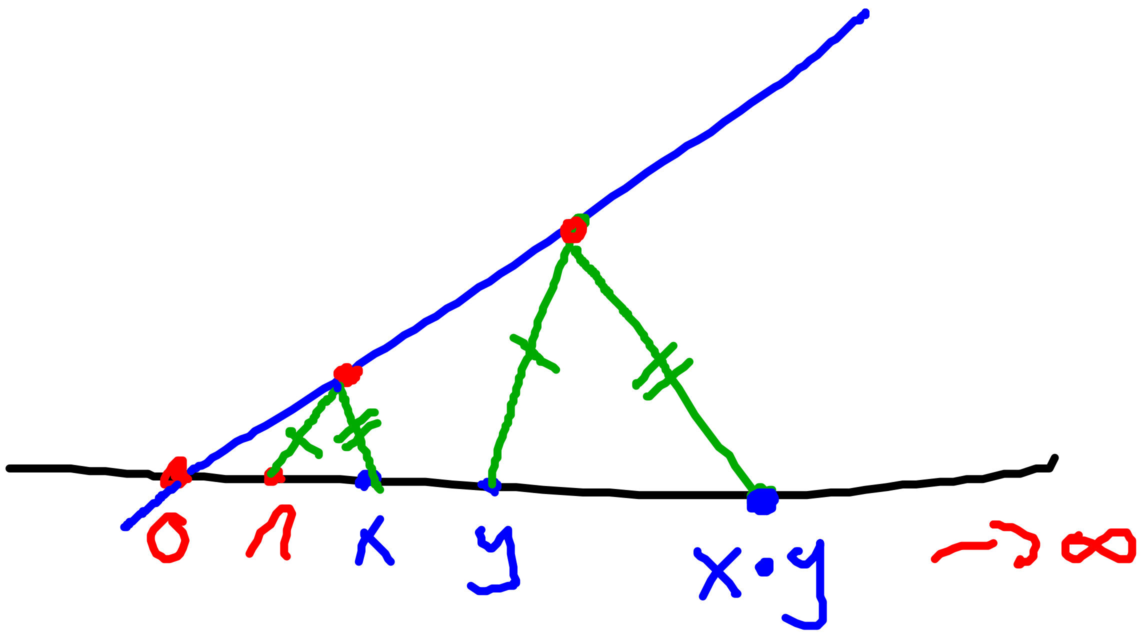
Last lecture: von Staudt Constructions.

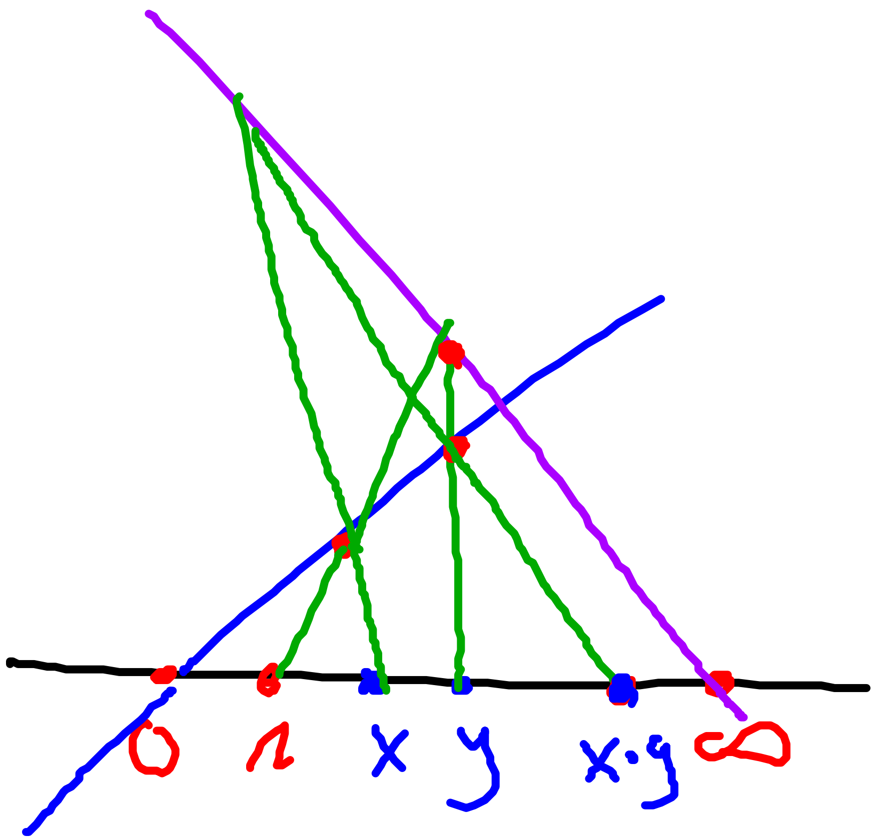
Addition





Multiplication



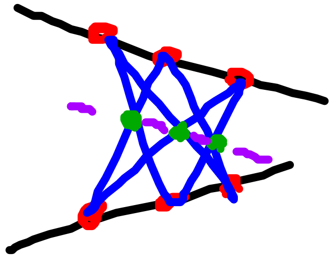


Topic today: How to extract field axioms from von Staudt constructions on Proj axioms

Starting point:

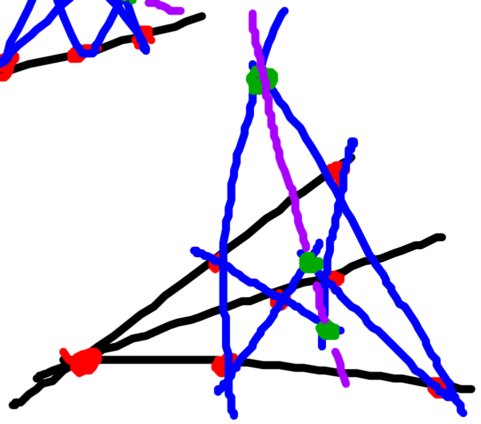
(A1, ..., A3) axioms of Projective Plane + certain Incidence theorems

(P) Thm of Pappos



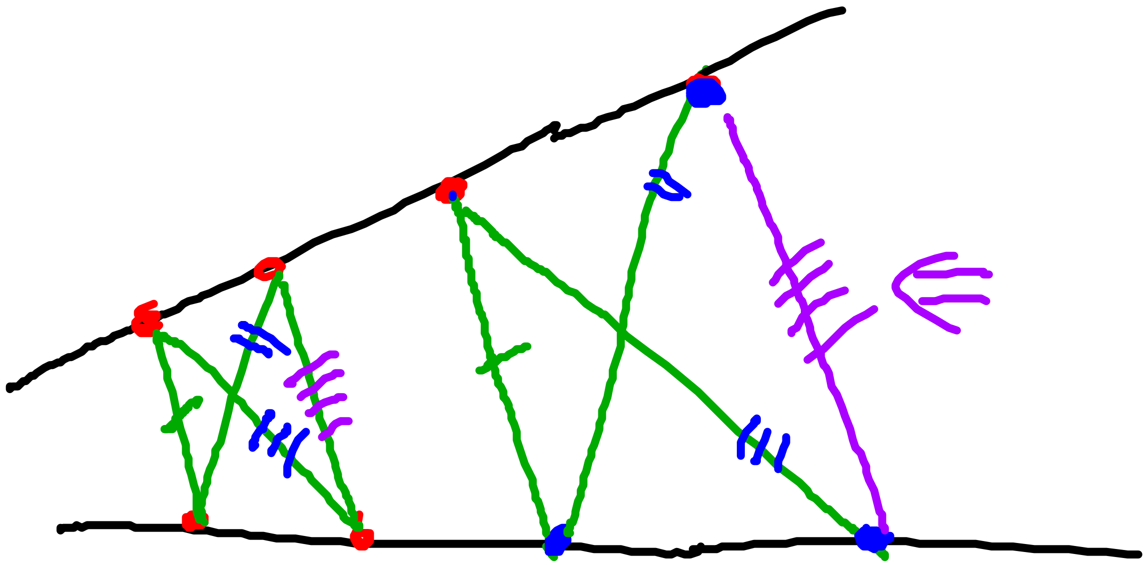
$\Downarrow$

(D) Thm of Desargues

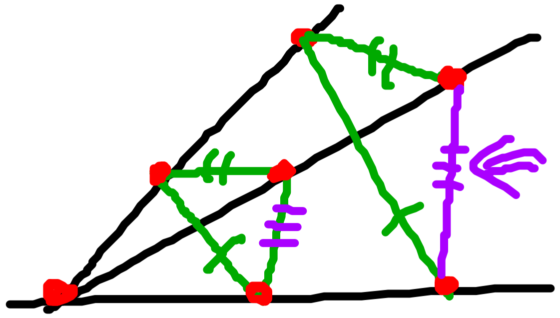


$\Updownarrow$

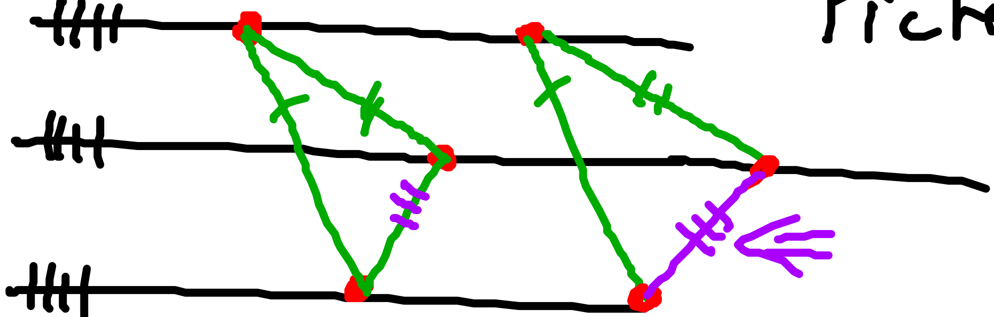
(S) „Scherensatz"



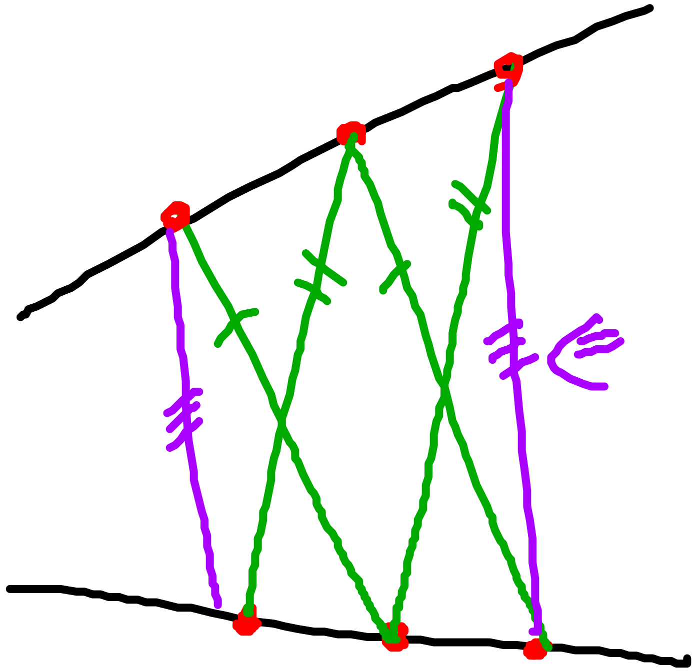
The „big" and the „little" Thms (in affine Pictures)



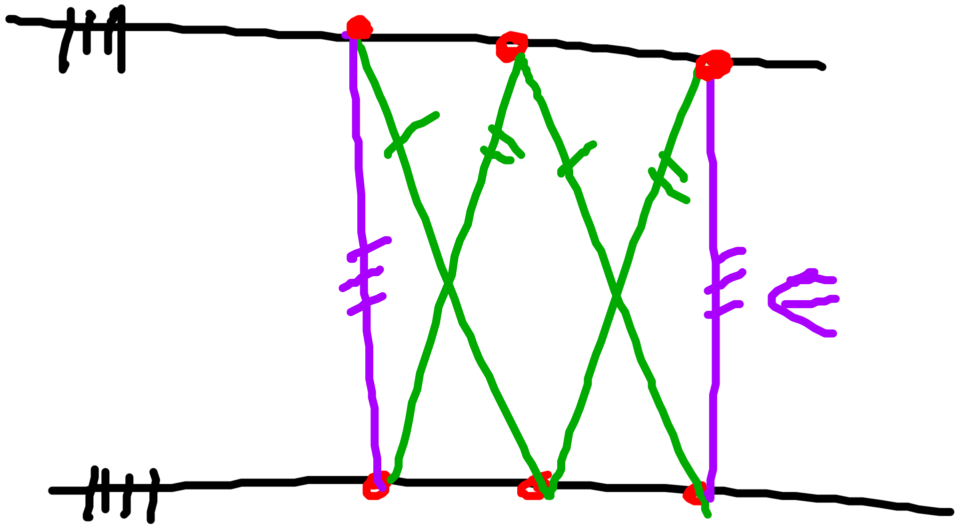
big Desargues (D)



little Desargues (d)
special case of (D)



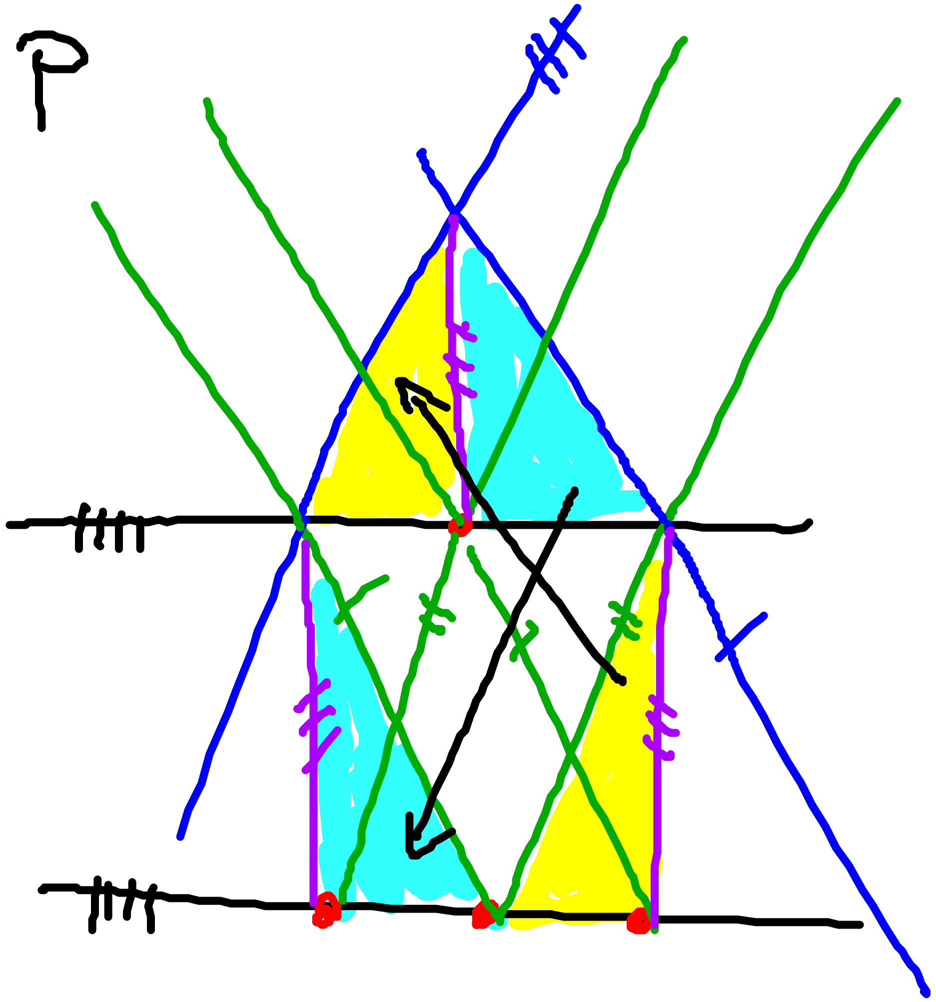
big Pappos (P)



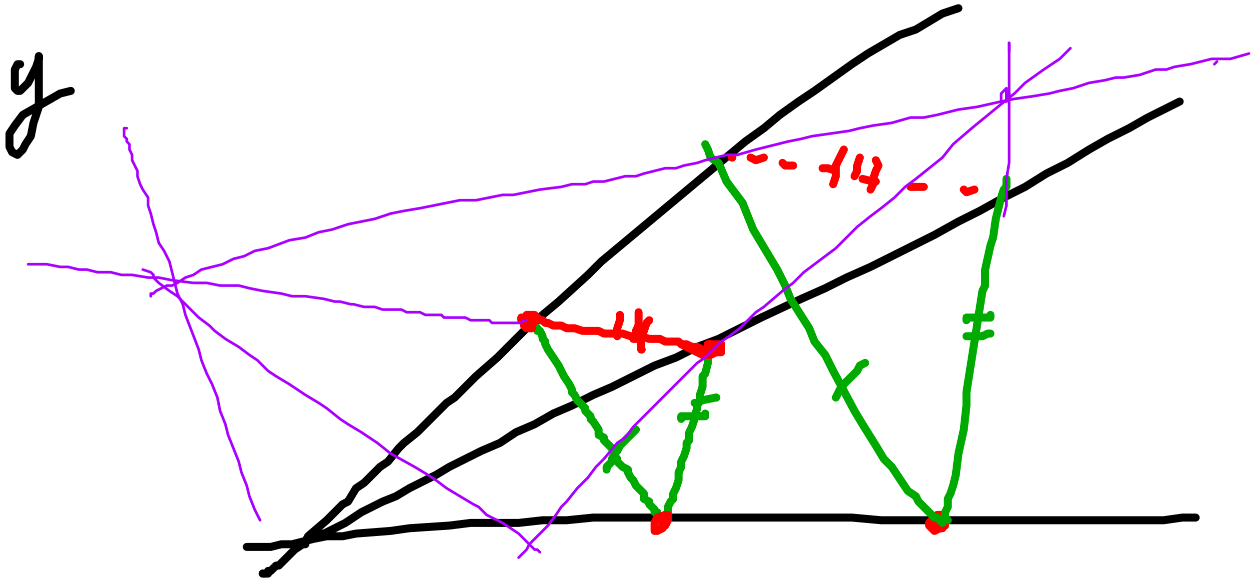
special case of (P)
little Pappos (p)

How are these theorems related

$$\begin{tikzcd}
P \arrow[r, Rightarrow, "\text{next slide}"'] & D \arrow[r, Rightarrow, "\text{special case}"] \arrow[d, Leftrightarrow, "\text{Exercises}"] & d \arrow[r, Rightarrow] & p \\
 & S & &
\end{tikzcd}$$



$$\underbrace{P \Rightarrow D}_{\text{Thm of Hessenberg}}$$



Field axioms from PG axioms + (P)

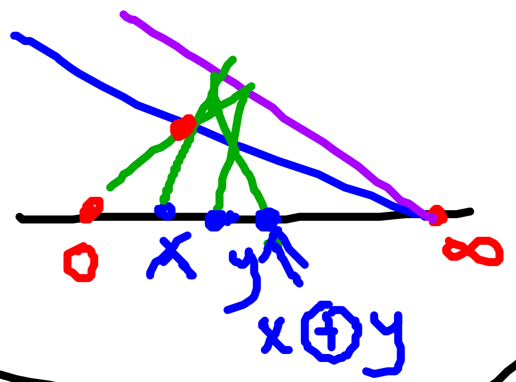
Assume A1 … A3 + P holds (then we also know D, S, d, p)

Single out a line ℓ on which we calculate

on that line fix three points $(0, 1, \infty)$

Consider $\mathbb{P} = \{p \mid p \, I \, \ell\} - \{\infty\}$

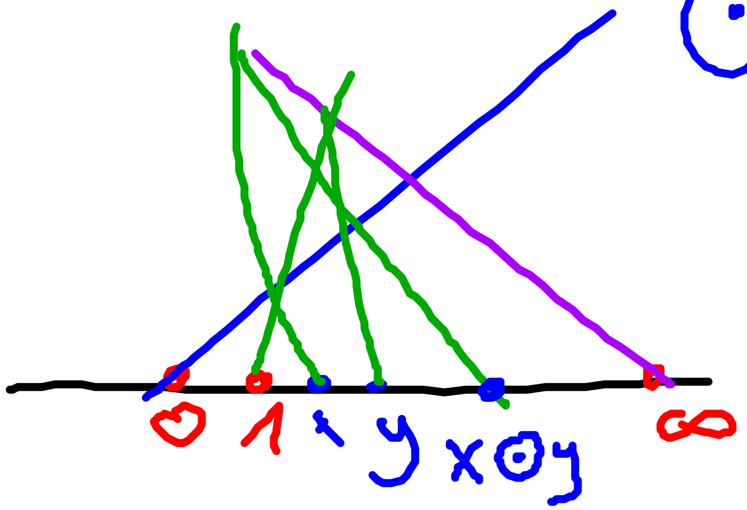
Define Addition



$\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$

Define Multiplication

$\odot : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$



Big aim

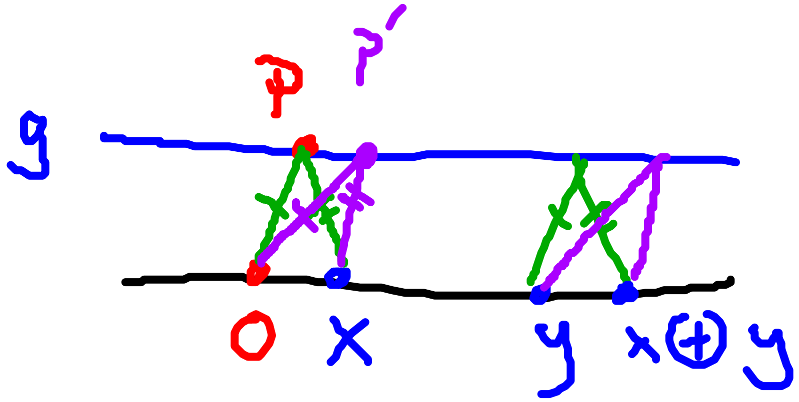
Prove that $(\mathbb{P}, \oplus, \odot)$ is a field

1. Step

show that $\oplus, \odot$ are well-defined

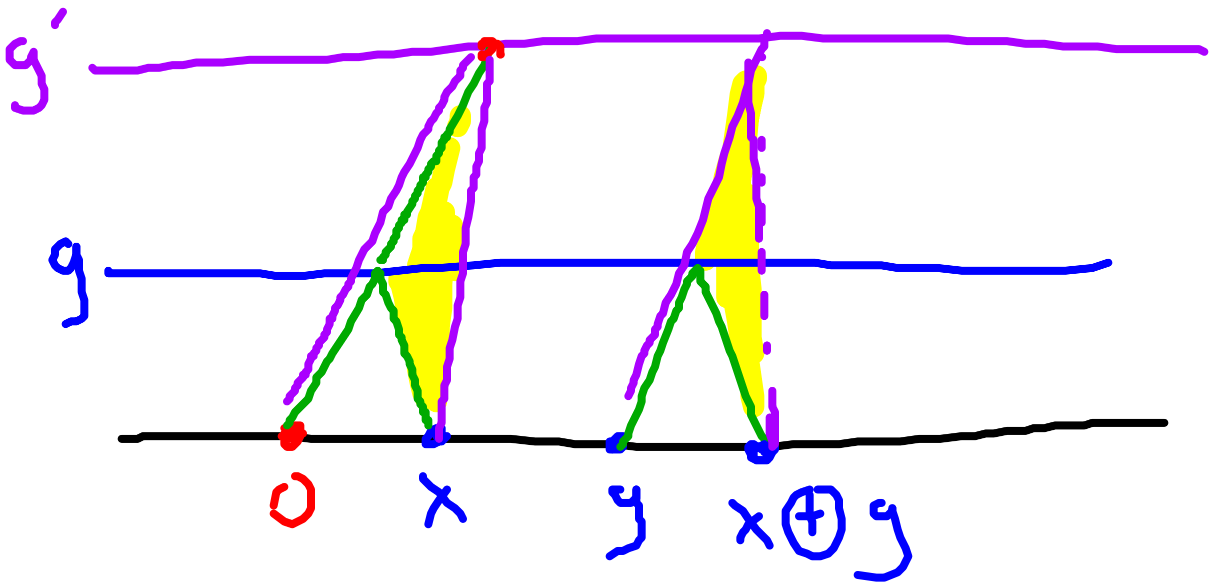
Example „$\oplus$" is well defined
(independent of the choice of auxiliary point)

1. P can be chosen everywhere on g



Ok. because of (S)

2.

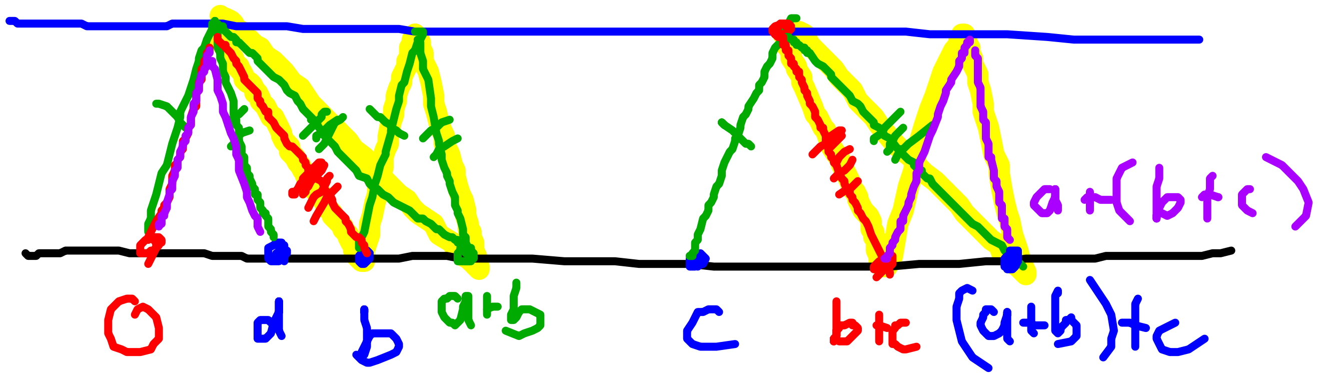


Ok. because of (d)

Show that $(\mathbb{P}, \oplus, \odot)$ is a field

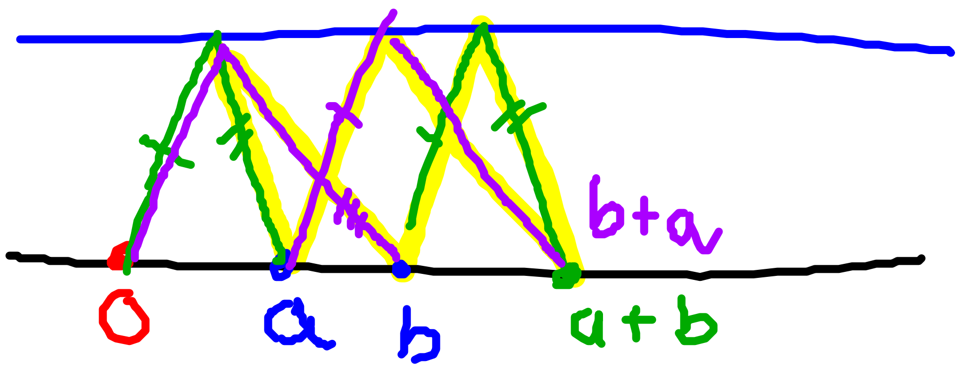
Many Axioms ....

For instance: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$



o.k. by (S)

$a \oplus b = b \oplus a$



o.k. by (P)

Similarly:

$a \cdot (b+c) = ab + ac$

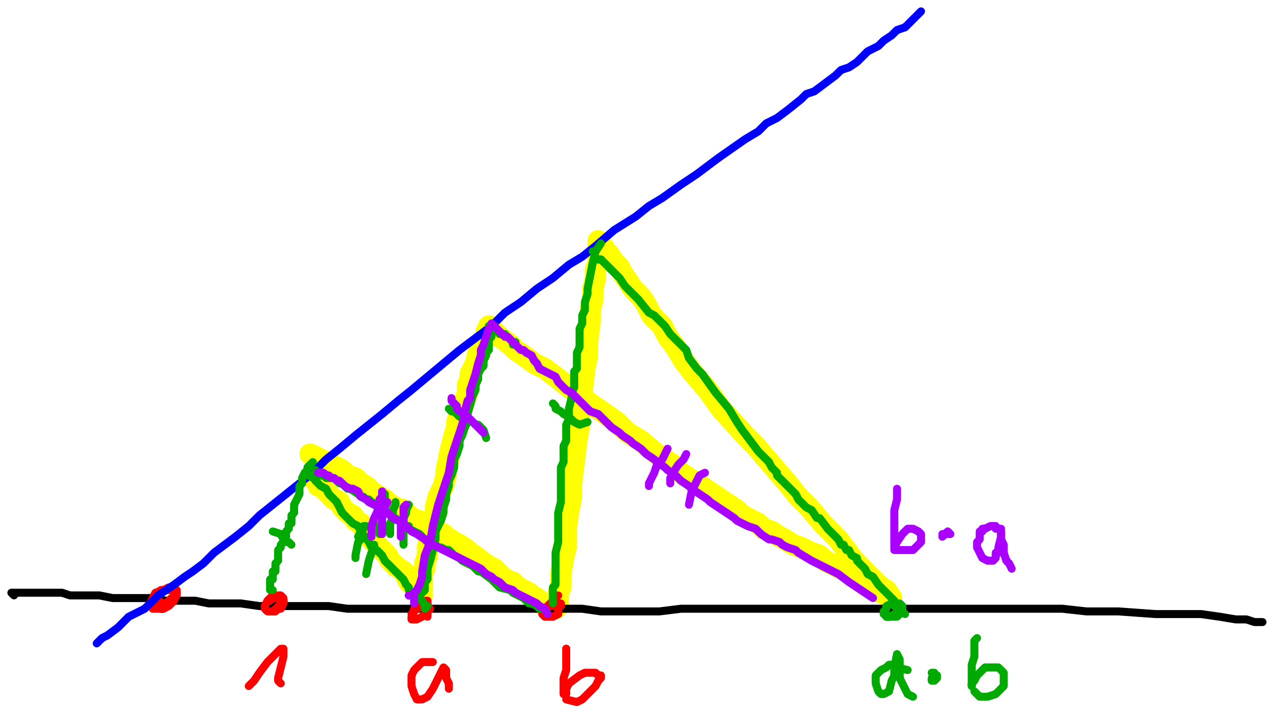
$(b+c) \cdot a = ba + ca$

$1 \cdot a = a$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

You can do all that by D, S

Still left: $a \odot b = b \odot a$

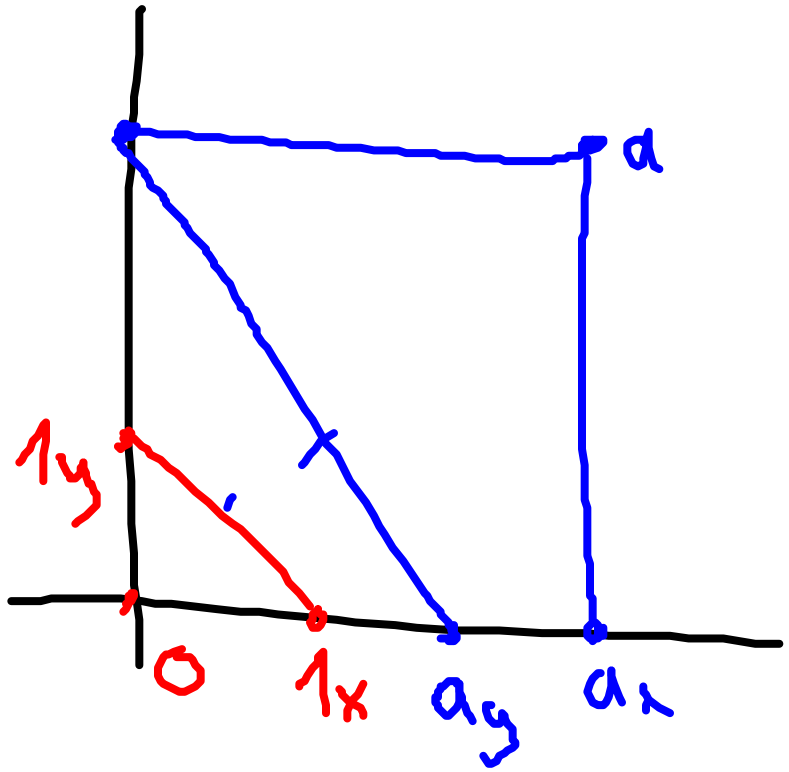


o.k. because of (P)

Thm: Our initial projective Plane is isomorphic to the proj Plane over $(\mathbb{P}, \oplus, \odot)$

$$\mathcal{P} = \frac{\mathbb{P}^3 - \{0\}}{\mathbb{P} - \{0\}}, \quad \mathcal{L} = \frac{\mathbb{P}^3 - \{0\}}{\mathbb{P} - \{0\}}$$

To prove:



Kollinearity of 3 Pts can be expressed by

$$\begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{vmatrix} = 0$$