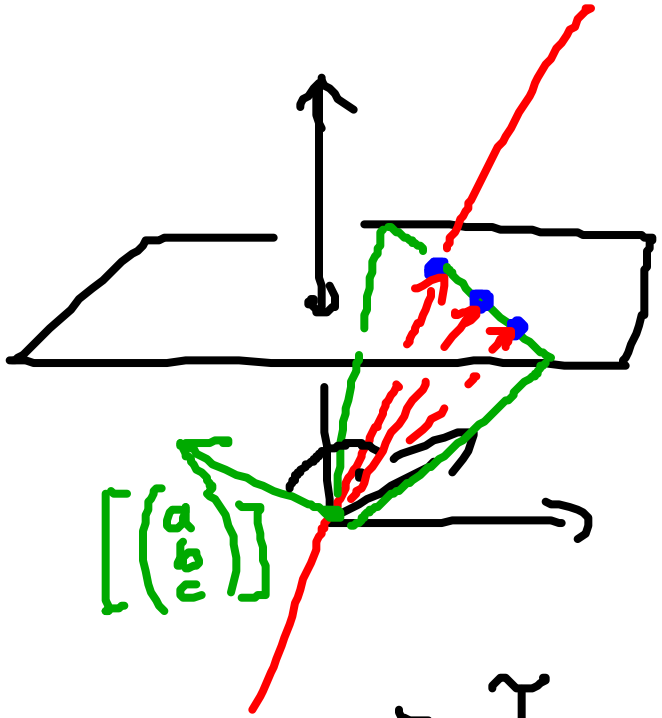
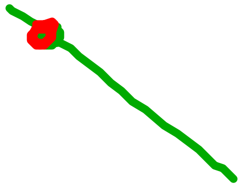
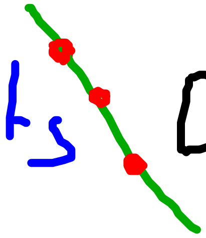
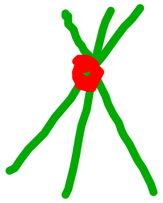
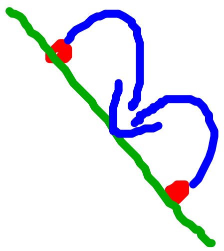
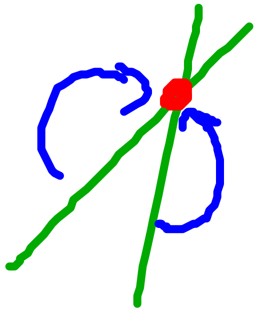
$\mathbb{R}^2 \to \mathcal{P}_{\mathbb{R}}$

$\begin{pmatrix}x\\y\end{pmatrix} \to \left[\begin{pmatrix}x\\y\\1\end{pmatrix}\right]$



- Incidence  $[p]\ I_{\mathbb{R}}\ [\ell] \Leftrightarrow \langle p, \ell \rangle = 0$
- collinearity of points  $[p], [q], [r]$ collin $\Leftrightarrow \det(p,q,r)=0$
- concurrence of lines  $[\ell], [g], [h]$ concurrent $\Leftrightarrow \det(\ell,g,h)=0$
- join  join $[p]\,[q]$ is $[p \times q]$
- Meet  Meet $[\ell]\,[m]$ is $[\ell \times m]$

Def: Collineation of a proj plane $(\mathcal{P}, \mathcal{L}, I)$ is a map $\tau : (\mathcal{P} \cup \mathcal{L}) \to \mathcal{P} \cup \mathcal{L}$; $\tau(\mathcal{P}) = \mathcal{P}$, $\tau(\mathcal{L}) = \mathcal{L}$ bijective $p \, I \, \ell \iff \tau(p) \, I \, \tau(\ell)$

Thm: Any collineation is already determined by the images of the points.

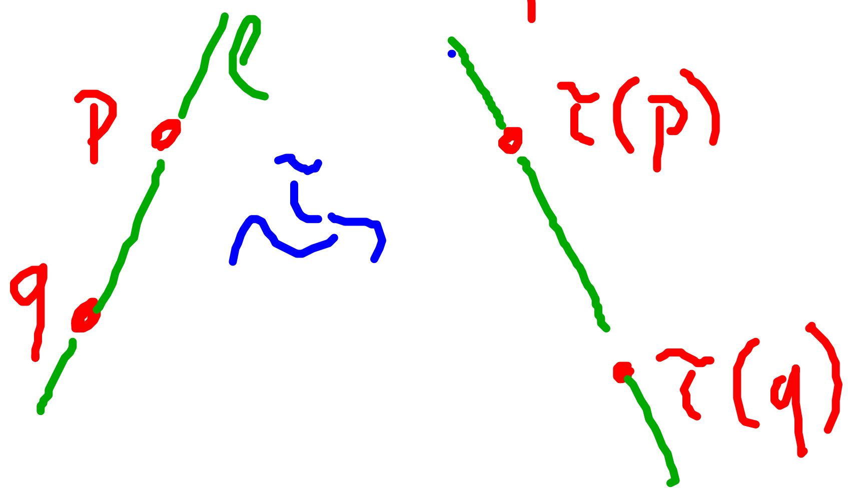
Proof: Let τ be a collineation, $\ell \in \mathcal{L}$

wlog $\ell = p \vee q$, $p \in \mathcal{P}$, $q \in \mathcal{P}$, $p \neq q$

$\Rightarrow p \, I \, \ell, q \, I \, \ell$

$\Rightarrow \tau(p) \, I \, \tau(\ell), \tau(q) \, I \, \tau(\ell)$

$\Rightarrow \tau(\ell) = \tau(p) \vee \tau(q)$



Def projective Transformation for $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$

Let $A \in \mathbb{K}^{3\cdot 3}$ be an invertable matrix

$$\tau_A\Big|_{\mathcal{P}_{\mathbb{K}}} : \mathcal{P}_{\mathbb{K}} \to \mathcal{P}_{\mathbb{K}}, \quad [p] \mapsto [Ap] \qquad \tau_A\Big|_{\mathcal{L}_{\mathbb{K}}} : \mathcal{L}_{\mathbb{K}} \to \mathcal{L}_{\mathbb{K}}, \quad [\ell] \mapsto [(A^{-1})^T \ell]$$

this is well defined

Thm: every projektive transformation is a collineation

Proof: τ_A is bijective since A was invertable

τ_A also preserves incidences:

$$[p]\, \mathcal{I}\, [\ell] \iff \langle p, \ell \rangle = 0 \iff p^T \cdot \ell = 0$$

$$\iff p^T \cdot E \cdot \ell = 0$$

$$\iff p^T \cdot \underbrace{A^T \cdot (A^T)^{-1}}_{} \cdot \ell = (A \cdot p)^T \cdot (A^{-1})^T \cdot \ell = 0$$

$$\iff [Ap]\, \mathcal{I}\, [(A^{-1})^T \ell]$$

For $(\mathcal{P}_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}}, \mathcal{I}_{\mathbb{R}})$ (the real proj. plane) every collineation is a proj trafo.

In general the following holds:

Every collineation of $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$ is the composition of a projektive trafo and a field automorphism.

$\varphi: \mathbb{K} \to \mathbb{K}$ bijective

$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$

$\varphi(a+b) = \varphi(a) + \varphi(b)$

Example

$\varphi: \mathbb{C} \to \mathbb{C}$

$z \to \overline{z}$

$\overline{a \cdot b} = \overline{a} \cdot \overline{b}$

$\overline{a+b} = \overline{a} + \overline{b}$

Def n-Points are in „general position"
if no three of them are collinear.

Thm In $(\mathbb{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, I_{\mathbb{K}})$ the following holds

let a, b, c, d be in general pos.

and a', b', c', d' be in " "

Then there exists a unique proj trafo mit
$\tau(a) = a'$, $\tau(b) = b'$, $\tau(c) = c'$, $\tau(d) = d'$

Proof: Go kalkule

You can assume $\underbrace{\text{w.l.o.g}}_{\text{up to proj Trafo}}$:

- one special point is an infinite Point
- two special points are infinite Points
- a special line is the line at Infinity
- four points in general pos have special coords.
- ⋮

Einschub: Incidence theorems and Proofstrategies

Incidence theorems are Thms that use <u>only</u> statements about points/lines/incidences.
(<u>no</u> circles, no conics, no lengths, no angels)

Structure of such Thms:

Hypotheses: Incidences: certain pts are on certain lines

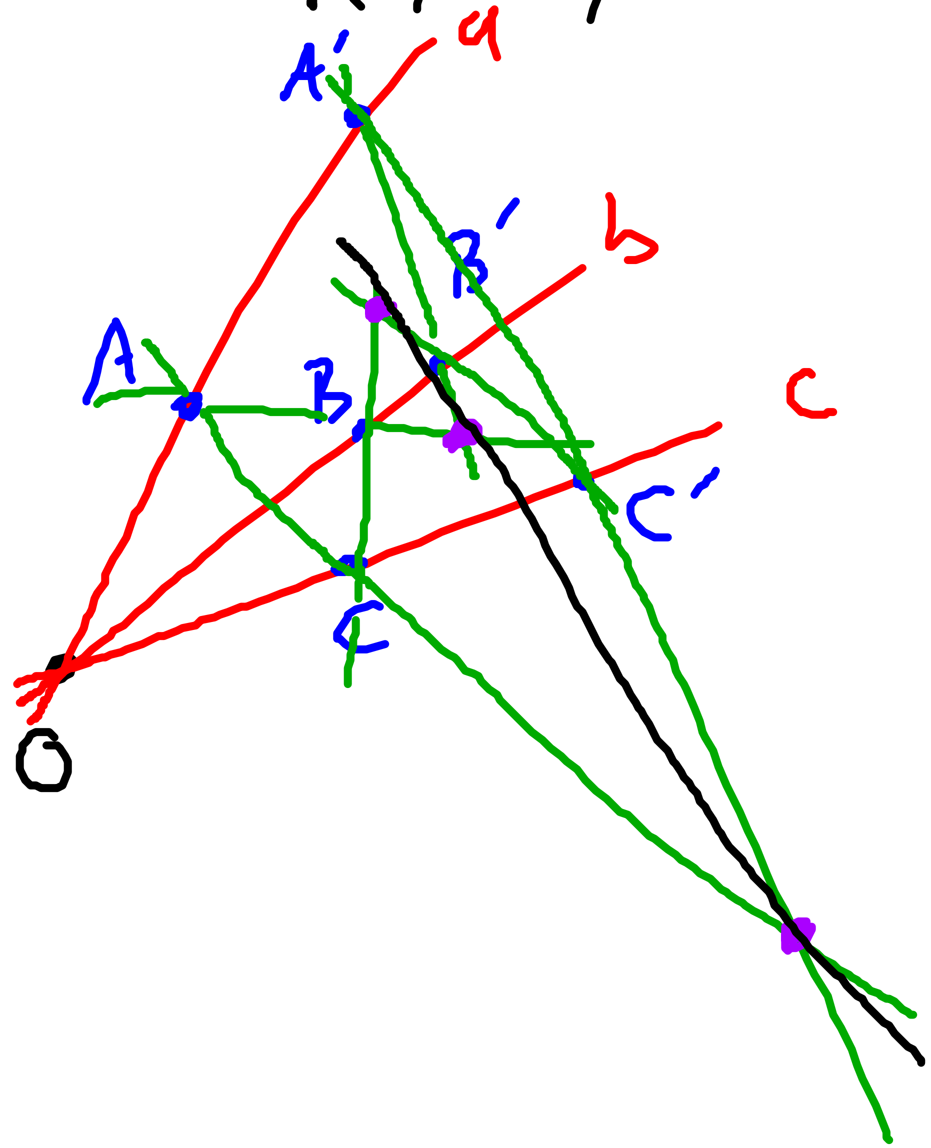
+ Nondegeneracy conditions: certain Incidences are <u>not</u> satisfied

Conclusion: some special incidence

Desargues' Theorem in $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I})$

Let a, b, c three different lines through a point O

Let A, A' be on a
B, B' be on b
C, C' be on c
these should be six different pts.



Then:

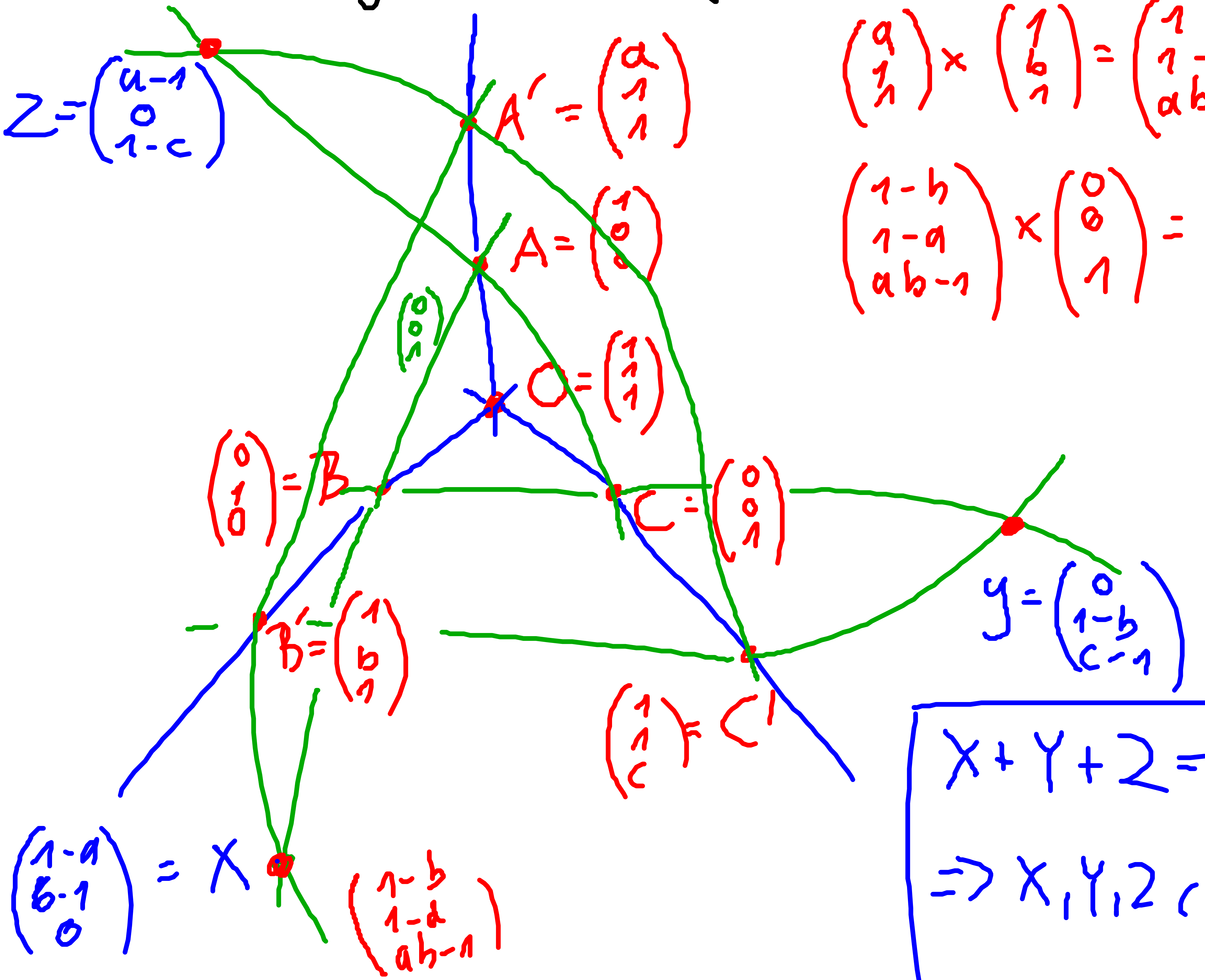
$$(A \vee B) \wedge (A' \vee B')$$

$$(B \vee C) \wedge (B' \vee C')$$

$$(C \vee A) \wedge (C' \vee A')$$

are kollinear!

1. Proof strategy: Brute force (with w.l.o.g. assumpt.)



$$\begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} = \begin{pmatrix} 1-b \\ 1-a \\ ab-1 \end{pmatrix}$$

$$\begin{pmatrix} 1-b \\ 1-a \\ ab-1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-a \\ b-1 \\ 0 \end{pmatrix} = X$$

$$\boxed{\begin{aligned} &X+Y+Z=0 \\ &\Rightarrow X, Y, Z \text{ collinear} \end{aligned}}$$