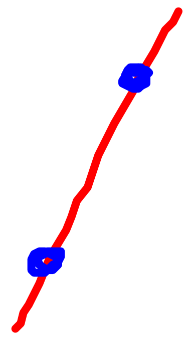
Axioms of Projective Planes $(\mathcal{P}, \mathcal{L}, I)$

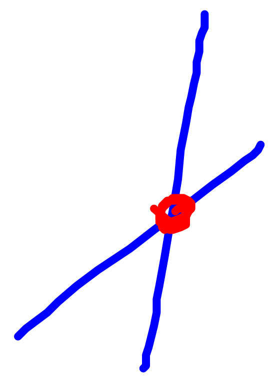
$\mathcal{P} \cap \mathcal{L} = \{\}$

$I \subseteq \mathcal{P} \times \mathcal{L}$

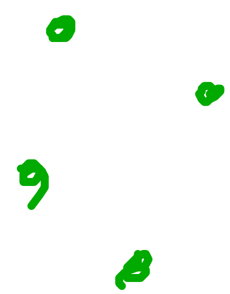
(A1) $\forall\, p, q \in \mathcal{P}$ with $p \neq q$
$\dot{\exists}\, \ell \in \mathcal{L}$ with $p \, I \, \ell$, $q \, I \, \ell$



(A2) $\forall\, \ell, m \in \mathcal{L}$ with $\ell \neq m$
$\dot{\exists}\, p \in \mathcal{P}$ with $p \, I \, \ell$, $p \, I \, m$



(A3) There ex. $a, b, c, d \in \mathcal{P}$
such that no three of them are collinear.



Def proj. Plane over a field $\mathbb{K}$

$$\mathcal{P}_{\mathbb{K}} = \frac{\mathbb{K}^3 - \{0\}}{\mathbb{K} - \{0\}}$$

$$\boxed{\begin{array}{l} x \in \mathbb{K}^3 - \{0\} \\ [x] = \{\lambda x \mid \lambda \in \mathbb{K} - \{0\}\} \end{array}}$$

$$\mathcal{L}_{\mathbb{K}} = \frac{\mathbb{K}^3 - \{0\}}{\mathbb{K} - \{0\}}$$

$\mathcal{I}_{\mathbb{K}}$: $[p] \in \mathcal{P}_{\mathbb{K}}$, $[\ell] \in \mathcal{L}_{\mathbb{K}}$

$$[p] \, \mathcal{I}_{\mathbb{K}} \, [\ell] :\Longleftrightarrow \langle p, \ell \rangle = 0$$

This is well defined

$\lambda \neq 0$, $\mu \neq 0$

$$0 = \langle p, \ell \rangle$$

$$\Longleftrightarrow$$

$$0 = \lambda \mu \langle p, \ell \rangle$$

$$= \langle \lambda p, \mu \ell \rangle$$

Thm For every field $\mathbb{K}$ the triple $(\mathcal{P}_{\mathbb{K}}, \mathcal{L}_{\mathbb{K}}, \mathcal{I}_{\mathbb{K}})$ is a projective Plane.

Proof: (A1)! Show that: $[p],[q] \in \mathcal{P}_{\mathbb{K}}$ with $\underbrace{[p] \neq [q]}_{p \neq \lambda q}$ there exists exactly one $[\ell] \in \mathcal{L}_{\mathbb{K}}$ with $\langle p, \ell \rangle = \langle q, \ell \rangle = 0$.

Search for non-trivial solutions of LES: $\underbrace{\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}}_{\text{rank } 2 \text{ since } [p] \neq [q]} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \dim(\ker(\dots)) = 1$

Take one solution ℓ and the corresp. class $[\ell]$

(A2) the same

(A3) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Finite fields $\Longrightarrow$ finite proj. planes

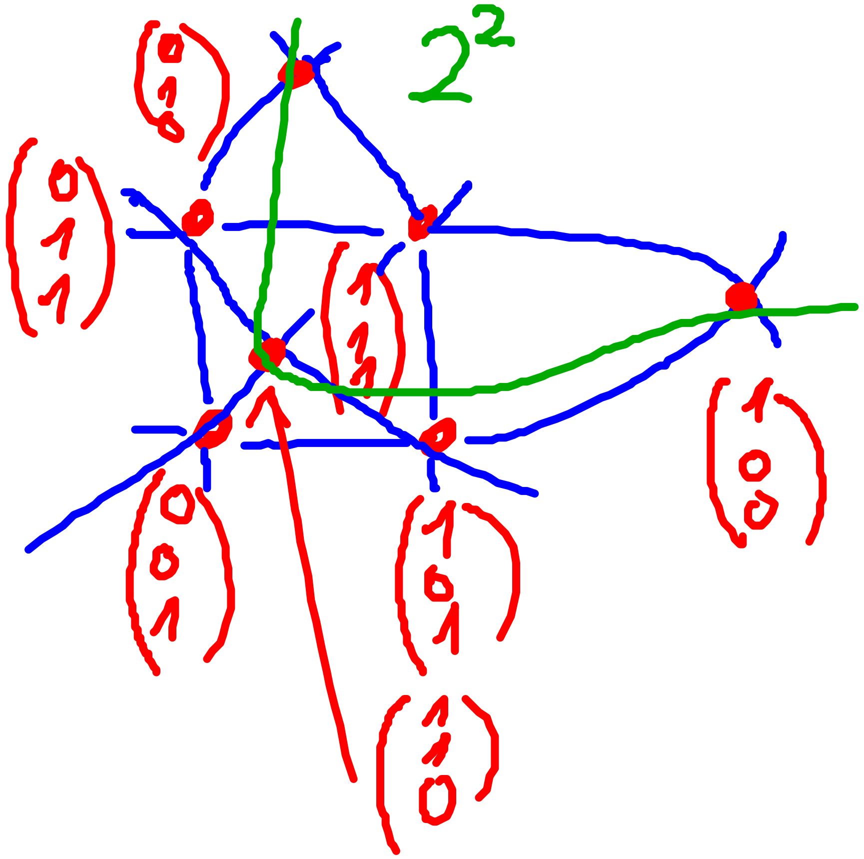
Example $GF_2 = (\{0,1\}, +, \cdot)$ calc. mod 2

Possible vektors in $(GF_2)^3 - \{0\}$

$$\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}}_{2^2}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}}_{2^1}, \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{1}$$

$$2^2 + 2^1 + 1 = 7$$

$$\boxed{[P] = \{P\}}$$

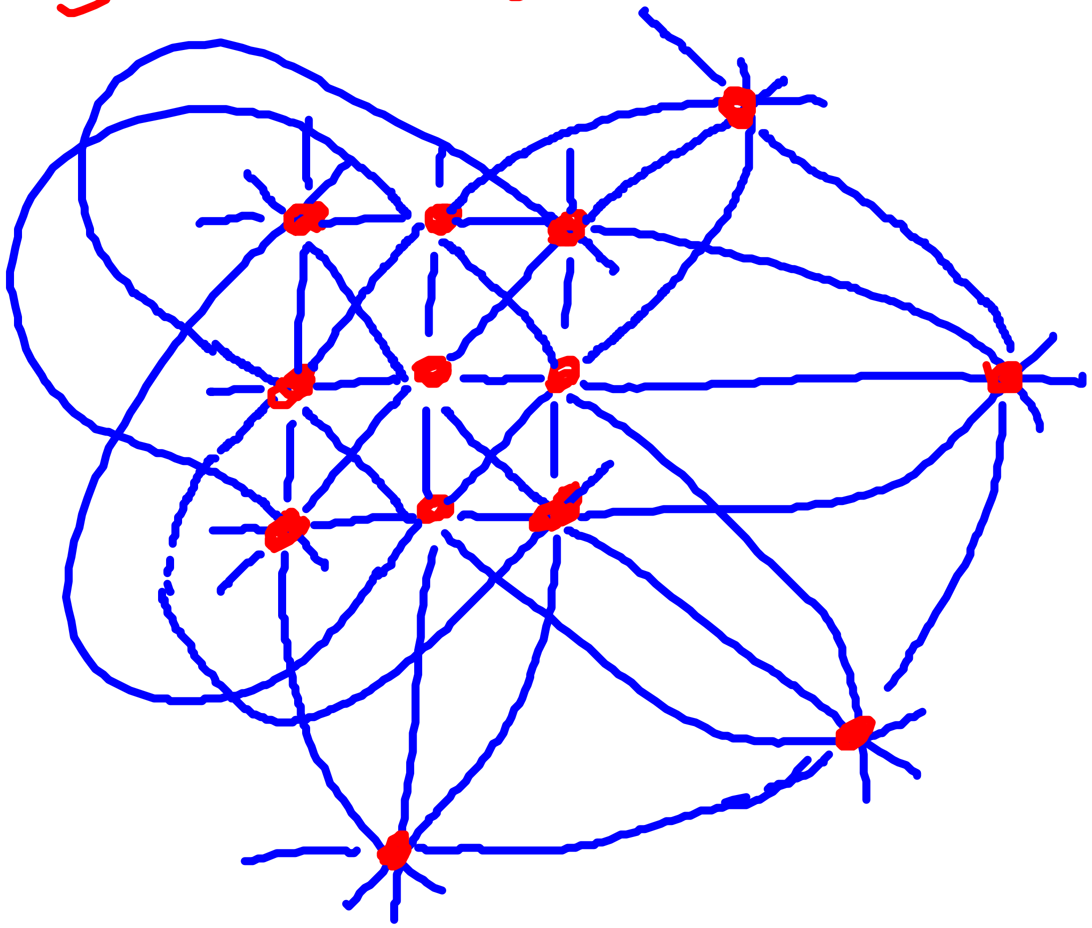


← Fano plane

Exp: $GF_3 = (\{0,1,2\}, +, \cdot)$ calc mod 3

Three types of Vektors

$$\underbrace{\begin{pmatrix} a_i \\ a_j \\ 1 \end{pmatrix}}_{3^2}, \quad \underbrace{\begin{pmatrix} a_i \\ 1 \\ 0 \end{pmatrix}}_{+\;3}, \quad \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{+\;1} = 13$$



Finite Fields:

$GF_n = \mathbb{Z}/_{n\mathbb{Z}} = \mathbb{Z}_n$

for n is prime

There also exist fields of the form $(GF_n)^k$

Consequence:

Thm: For every prime power q there is a Proj. Plane of order q.

Conjecture:

The order of any projective plane is a prime power

Thm of Bruck & Ryser (~ 1949)

If the order n of a proj plane is $\equiv 2 \mod 4$ or $\equiv 1 \mod 4$ then $n = i^2 + j^2$ for suitable $i, j \in \mathbb{N}_0$

Proof: Do it yourself

what does this mean:

70s 2000 CPU hours on a Cray

Order: 2 3 4 5 ~~6~~ 7 8 9 ~~10~~ 11 12

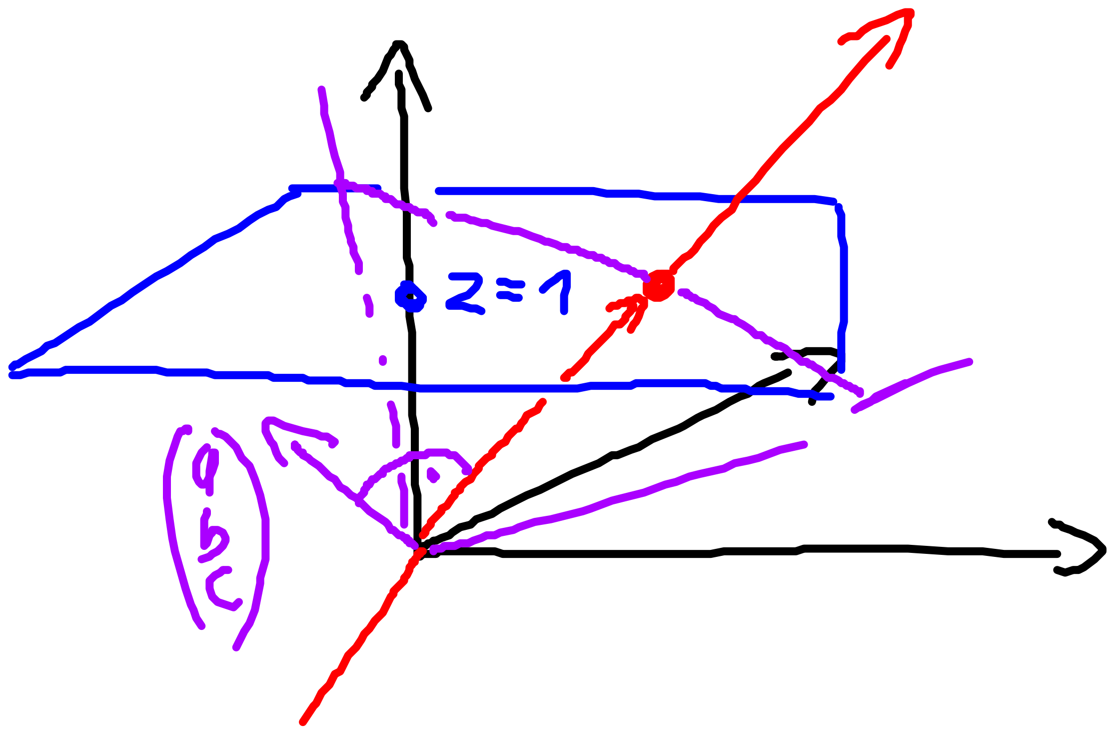
2	3	4	5	6	7	8	9	10	11	12
GF_2 Fano plane $2=1+1$	GF_3	$(GF_2)^2$	GF_5 $5=1+4$	BR	GF_7	$(GF_2)^3$	$(GF_3)^2$ $9=9+0$	$10=9+1$	GF_{11}	still open

Geometric Interpretation over $\mathbb{R}$

$\mathbb{R}^2$ is the „usual" number plane (Euclidean plane)

Homogenize:

$$\begin{pmatrix}x\\y\end{pmatrix} \to \left[\begin{pmatrix}x\\y\\1\end{pmatrix}\right]$$



$\left[\begin{pmatrix}x\\y\\0\end{pmatrix}\right]$ Points at Infinity

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Lines $\left[\begin{pmatrix}a\\b\\c\end{pmatrix}\right]$

$\begin{pmatrix}a\\b\\c\end{pmatrix}$ Normal vector of a plane which intersect the „drawing plane" at a corresponding line.