Pseudoline arrangements — a brief overview

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Abstract

Dear reader, what you are going to read is a revision of my presentation about pseudoline arrangements on 24th July 2010. In particular I want to thank Jürgen Richter-Gebert for his support to make the topic accessible to me.

However in this seminar I focused on three aspects of pseudoline arrangements:

1. General properties of the latter as a representation of oriented matroids will show the basics.
2. Mutations of simplicial cells and their applications and the topic of realization are results which will appear more plausible thanks to the general ideas with regard to the general properties.

After reading my article you should have got a general idea about the basic aspects on pseudoline arrangements. By the beginning of 2011 there should be a more extensive version available as I will have expanded the essay to my bachelor’s thesis.

1. Basics about pseudoline arrangements

1.1. Introduction: The surface

First of all I want to hand you some practical descriptions of the term of a pseudoline arrangement. The main feature of those is that they are universal and may universally be applied. I want to give you the pseudoline image of a rank 3 matroid: At first you see a collection of deformed lines in a circle here. Each pseudoline may be represented as an image of a line in \( \mathbb{R}P^2 \) by a homeomorphism from \( \mathbb{R}P^2 \) onto itself. I want to stress that there is no related representation to each collection of pseudolines in \( \mathbb{R}P^2 \) as space of all lines. Despite that obvious need for distinction many properties of such line arrangements apply to pseudoline arrangements as well. This is to say: There isn’t necessarily a corresponding line arrangement to every pseudoline arrangement. This will later be discussed in the section realizability.

As in \( \mathbb{R}P^2 \) lines are closed curves cutting \( l_\infty \) so we may identify the ends of the lines. Secondly it is characteristic to pseudoline arrangements that two pseudolines would meet exactly once. Notably a circle may surround the pseudoline arrangement which is a common practice when drawing the latter. Remark that this circle may represent a pseudoline as well. The related element in the \( \mathbb{R}P^2 \) as space of all lines is commonly \( l_\infty \) in this case.

There are some further ways to look at pseudoline
arrangements.

1.2. The matroid view

Pseudoline arrangements may be viewed as a realization of oriented matroids. I assume you are familiar with these. Elsewise you might have a look at the essay of Sebastian Jansen who held the presentation about the matroid basics in the seminar. In the following I will refer to the matroid representation $M = (E, L)$, where $E$ shall represent the set of pseudolines and $L$ shall be the set of covectors created by $E$. Especially this set includes elements representing all intersection points of two or several pseudolines, any line sections and all the areas in the set.

Note that two certain areas adjacent to a virtual surrounding pseudoline (e.g. a pseudoline created for the purpose of demonstration, not belonging to the arrangement) may be identified. This may be the case if the ultimate edges (those who intersect with the virtual surrounding pseudoline on the closure of the areas) do belong to the same pair of lines. For example you might draw a simple triangle and expand the sides to lines. Each of the opposing areas (apparently a quad and a triangle) form a triangle sided by all three lines. The view of the matroid as pseudo great circle arrangement will provide deeper insights.

How is this represented in the set of covectors then?

The set $E = \{l_1,l_2,...,l_n\}$ contains the $n$ oriented pseudolines representing the matroid. Any of the covectors in $L$ has the form $(\sigma_1,\sigma_2,...,\sigma_n)$ while $\sigma_i \in \{-,0,+,0\}$ for all $i = 1,2,...,n$. Here $\sigma_i$ signifies the relation of the object (point, line section or area) to the $k$-th oriented pseudoline $l_k$. If the object is on the positive side of $l_k \sigma_k$ is assigned $+$. In contrary we assign $-$. Obviously if the object is found on $l_k$ we assign $\sigma_k = 0$.

It can be shown that $M = (E, L)$ will fulfill the covector axioms for oriented matroids. The axiom demanding for contrarily signed covectors is rather of theoretical concern. The proceeding approach to pseudoline arrangements will illustrate why we can accept this.

1.3. The pseudo great circle view

As we know (oriented) hyperplane arrangements of rank 3 are representations of (oriented) matroids of rank 3. Without limit we may intersect with the $S^2$ sphere. This results in an image as below.

By central projection of the sphere and its great circles we receive the desired $\mathbb{RP}^2$-image of the hyperplane arrangement.

Vice versa pseudoline arrangement correspond to pseudoplane arrangements. Those do not necessarily correspond to hyperplane arrangements.

In particular we can represent pseudolines in $\mathbb{RP}^2$ as pseudo great circles on the $S^2$ sphere. A single (oriented) pseudo great circle is to be imagined as follows:

This stresses two things which have been assumed before: It makes sense to identify antipodal points. And it makes sense to allow triangles being split by $l_\infty$ all the while - as long as $l_\infty$ is none of the pseudolines itself!

The arbitrary choice of the visible covector sign code in the pseudoline arrangement is lined out here: If $l_\infty$
is not part either a covector or its negative will be to the front of the sphere - fully depending from your angle.

1.4. Zonotopes and pseudoline arrangements

Another representation of oriented matroids are (signed) zonotopes. Jan Erik Müller had an own presentation on the topic. Concerning my field I want to line out that there is a very simple translation between zonotopes and pseudoline arrangements. We know that zonotopes are spanned by a bunch of vectors. Likewise the matroid of pseudoline arrangements is spanned on pseudolines. The covector sign codes are the same relative to its basic set. Indeed flipping the roles of intersection points and areas does the job!

So pseudoline arrangements are one of many representations of oriented matroids. \( \chi \) symbols the set of covectors of the matroid \((\sigma_1, \sigma_2, \ldots, \sigma_n)\).

2. Mutations and simplicial cells

The second topic I want to cover is the topic of mutations. As for zonotopes this is called a flip. In a pseudoline arrangement the simplicial cell – a triangle – is flipped by pushing a pseudoline over the two others who close on the triangle. It comes natural that those covectors belonging to the flipped triangle (3 points, 3 edges, 1 area) are the only ones to be affected by this simplicial mutation. In effect the signs of the the involved covectors refering to the involved pseudolines are reversed. In a way this makes invisible covectors visible (on pseudogreat circles: brings them to the front), while others (those of the unflipped triangle) are dismissed from the (visible) pseudoline arrangement.

In particular this remains consistent with the covector axioms.

2.1. Basic properties

Las Vergnas stated in 1980 that two arbitrary pseudoline arrangements with the same number of lines may be deformed to one another by such a series of simplicial mutations. The conjecture is proved true for any rank 3 matroids, e.g. pseudoline arrangements embedded in \( \mathbb{RP}^2 \).

Basically triangles are the most interesting element of a pseudoline arrangement. Research is focusing on these simplicial cells. Basic results are for example:

If \( n \) is the number of pseudolines of an oriented matroid the minimum number of triangles in the realized (matroid) is \( n \).

Another one is:

If \( n \) is at least 4 the maximum number of triangles in a arbitrary realized matroid is \( \frac{n(n-1)}{3} \).

2.2. Higher dimensions

Matroids of rank 4 yield a different simplicial cell, which is topologically equivalent to a tetrahedron. Roudneff proved for example that a realization of 8 pseudohyperplanes may contain 7 simplicial regions only.

A spectacular result of Richter-Gebert states that if the number of pseudohyperplanes \( n \) is at least 20 there may exist a pseudohyperplane which is not adjacent to any simplicial region. This remarkable as there is no pseudoline arrangement with a similar property.

3. Realizability

3.1. Basic facts

For illustration of realizability we will employ a simple and popular counter-example: The so called non-Pappus - a deformed Pappus configuration. The Pappus Theorem is an incidence configuration which would force \( l \) to intersect \( p \) in the following image in case of a line arrangement realization. Thus the specific covector of the matroid could not be realized. So pseudoline arrangements are the solution to realize this chirotope.
Another condition for realizability of chirotopes resembles a Grassmannian-Plücker-relation. This is for good reason as the realization in $\mathbb{RP}^2$ requires that the relation could possibly be solved by (correctly signed) figures. Rank 3 chirotopes where
\[
\{ \chi(e,a,b) \times \chi(e,c,d), \\
-\chi(e,a,c) \times \chi(e,b,d), \\
\chi(e,b,c) \times \chi(e,a,d) \}
\]
includes $\{-1, +1\}$ or neither are realizable in terms of pseudoline arrangements.
A crucial theorem is the one of Folkman Lawrence in 1978:

Any rank 3 oriented matroid is realizable as pseudoline arrangement in $\mathbb{RP}^2$.

While for rank 2 oriented matroids this is obviously true (we realize points in $\mathbb{RP}^1$) similar applies to matroids of higher rank.

The above theorem leads us to a further description of realizability:
While any rank 3 oriented matroid can be realized by a pseudoline arrangement the latter could at times vividly be stretched to a line arrangement (while chirotope properties are maintained). Realizability does occur as stretchability!

3.2. Approaches by computational engineering

This theorem motivated the development of algorithms. For example Richter-Gebert contributed to that by realizing of chirotopes by sequences of extensions to partly realized line arrangements. Particularly interesting is the principle to operate in the dual space of the lines: Vice versa to line realizations a mutation is a flip of a point over the join of two other points! All the same this remains a special approach to realization. So while there were important improvements the question of realizability can not be answered ultimately by algorithms by now. Science is working to improve, though.

For more information I advise you Kombinatorische Realisierbarkeitskriterien für orientierte Matroide and New construction methods for oriented matroids by Jürgen-Richter Gebert which will serve with more detailed explanations. They served as resource for my presentation, too. Some pictures have also been taken from "Oriented Matroids" by Björner, Las Vergnas, Sturmfels, White and Ziegler.