11

Calculating with conics

A major part of classical elementary geometry was concerned with the question which constructions can be carried out by a ruler and a pair of compasses. The decisive primitive operations there are connecting two points by a line, drawing a circle with a radius given by two other points and marking intersections of objects as new points. In our projective framework we do not have circles, but still can consider elementary constructions with the objects we have studied so far (points, lines and conics). Since we are interested in particular in *calculating* with geometric objects we in particular want to know how we can *compute* the results of geometric primitive operations if the parameters of the involved objects are given. So far we can roughly associate the classical ruler/compass operations with our projective operations in the following way:

- connecting two points by a line corresponds to the *join* operation and can be carried out by a cross product.
- intersecting two lines corresponds to the *meet* operation and is also carried out by the cross product
- constructing a circle from two points can be associated to our construction of a *conic by five points* as described in Section 10.1.

Furthermore we had additional operations for

- constructing polars of points and lines w.r.t. a conic (this included the calculation of a tangent),
- transforming points/lines/conics by projective transformations,
- calculating the matrix of the dual of a given conic.

So far we do not have an equivalent the the classical operations of intersecting a line and a circle or for intersecting two circles. This chapter is (among others) dedicated to this task. We will develop algebraic methods for intersecting conics with lines and conics with conics. Clearly these operations can be carried out by solving corresponding systems of polynomial equations. However, we will try to make the calculations for these operations as natural as possible in the framework of homogeneous coordinates and matrix representations for conics. This chapter is meant as a collection of cooking recipes for such kinds of primitive operations.

11.1 Splitting a degenerate conic

Before we will turn our attention to the problem of intersecting a conic with other objects we will study how it is possible to derive homogeneous coordinates for the two lines of a degenerate conic from the matrix of a conic. This will turn out to be a useful operation later on.

Assume that a conic $\mathcal{C}_A$ is given by a symmetric matrix A such that as usual $\mathcal{C}_A = \{p \mid p^T A p = 0\}$. If the conic is degenerate and consists of two lines or of one double line then A will not have full rank. Thus we can determine a degenerate situation by testing $\det(A) = 0$. If a conic consists of two distinct lines with homogeneous coordinates g and h then its symmetric (rank 2) matrix A can be written as $A = gh^T + hg^T$ (up to a scalar multiple). The matrices gh^T and hg^T would in principle generate the same conic, but they are not symmetric. However, knowing one of these matrices (for instance gh^T) would be equivalent to knowing the homogeneous coordinates of the two lines, since the columns of this matrix are just scalar multiples of h and the rows are just scalar multiples of g. Any non-zero column (resp. row) could serve as a homogeneous coordinates for g (resp. h). This can be seen easily by observing that the disjunction of the conditions $\langle p, h\rangle = 0$ and $\langle p, g\rangle = 0$ can be written as

$$0 = \langle p, g\rangle \cdot \langle p, h\rangle = (p^T g)(h^T p) = p^T (gh^T) p.$$

So, splitting a degenerate conic into its two lines essentially corresponds to finding a rank 1 matrix B that generates the same conic as A. The quadratic form is linear in the corresponding matrix (i.e. we have $p^T(A+B)p = p^T Ap + p^T Bp$). Furthermore those matrices for which the quadratic form is identically zero are exactly the skew symmetric matrices (those with $A^T = -A$). All in all for decomposing a symmetric degenerate matrix A into two lines we have to find a skew symmetric matrix B such that $A + B$ has rank 1. Thus in our case we have to find parameters λ, μ and τ such that the following matrix sum has rank 1

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} + \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix}.$$

The determinant of every 2×2 sub-matrix of a rank 1 matrix must vanish. Thus necessary conditions for the parameters are:

$$\begin{vmatrix} a & b+\tau \\ b-\tau & c \end{vmatrix} = 0; \quad \begin{vmatrix} a & d-\mu \\ d+\mu & f \end{vmatrix} = 0; \quad \begin{vmatrix} c & e+\lambda \\ e-\lambda & f \end{vmatrix} = 0.$$

resolving for λ, τ and μ gives:

$$\tau^2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix}; \quad \mu^2 = \begin{vmatrix} a & d \\ d & f \end{vmatrix}; \quad \lambda^2 = \begin{vmatrix} c & e \\ e & f \end{vmatrix}.$$

This determines the parameters λ, μ and τ up to their sign. In principle one could test all eight possibilities to find a suitable choice that makes the entire matrix a rank 1 matrix. However, there is also a more direct way of calculating the values with their correct sign. For this we associate to the parameter vector $p = (\lambda, \mu, \tau)^T$ the skew-symmetric matrix

$$\mathcal{M}_p := \begin{pmatrix} 0 & \tau & -\mu \\ -\tau & 0 & \lambda \\ \mu & -\lambda & 0 \end{pmatrix}.$$

Left multiplication with the matrix $\mathcal{M}_p$ encodes performing a cross-product with the vector p. A simple calculation shows that for any three dimensional vector q we have

$$\mathcal{M}_p \cdot q = p \times q.$$

Lemma 11.1. *Let A be a rank 2 symmetric 3×3 matrix that defines a conic consisting of two distinct lines. Let p be the point of intersection of these lines. Then for a suitable factor α the matrix $A + \alpha\mathcal{M}_p$ has rank 1.*

Proof. Let g and h be homogeneous coordinates for the two lines. We may assume that these coordinates are scaled such that A has the form $gh^T + hg^T$. The intersection of these lines is $g \times h$. We have for a suitable factor α the equation $g \times h = \alpha p$. If we consider the difference $gh^T - hg^T$ we obtain the skew symmetric matrix

$$gh^T - hg^T = \begin{pmatrix} 0 & g_1h_2 - g_2h_1 & g_1h_3 - g_3h_1 \\ g_2h_1 - g_1h_2 & 0 & g_2h_3 - g_3h_2 \\ g_3h_1 - g_1h_3 & g_3h_2 - g_2h_3 & 0 \end{pmatrix}.$$

Comparison of coefficients shows that this matrix is nothing else as $\mathcal{M}_{g\times h}$. Thus we obtain

$$gh^T - hg^T = \mathcal{M}_{g\times h} = \mathcal{M}_{\alpha p} = \alpha\mathcal{M}_p$$

With this we obtain:

$$A - \alpha\mathcal{M}_p = (gh^T + hg^T) + (gh^T - hg^T) = 2gh^T.$$

Thus the result must have the desired rank 1 form and the theorem is proved. □

In particular if for specific coordinates g and h we have $A = gh^T + hg^T$ and $p = g \times h$ we can chose the factor $\alpha = 1$ and obtain

$$A - \mathcal{M}_p = 2gh^T.$$

If we instead add the matrices we obtain $A - \mathcal{M}_p = 2hg^T$. The last lemma allows us to calculate the corresponding rank 1 matrix if a symmetric matrix of a degenerate conic is given. However, it has one big disadvantage: We do neither know p nor α in advance. There are several circumstances (we will encounter them later) in which we, for instance, know p in advance. However it is also possible to calculate p from the matrix A without too much effort. For this we have to use the formula

$$(gh^T + hg^T)^\triangle = -(g \times h)(g \times h)^T.$$

which we already used in Section 9.5. The the matrix $B = (g \times h)(g \times h)^T$ is a rank 1 matrix pp^T with $p = g \times h$. Thus each row/column of this matrix is a scalar multiple of p. Furthermore the diagonal entries of this matrix correspond to the squared coordinate values of $g \times h$. Thus one can extract the coordinates of $g \times h$ by searching for a non-zero diagonal entry of B, say $B_{i,i}$, and set $p = B_i/\sqrt{B_{i,i}}$ where B_i denotes the i-th column of B. With this assignment we can calculate the matrix $A^\triangle + \mathcal{M}_p$. Which gives either $2gh^T$ or $2hg^T$ depending on the sign of the square root. All together we can summarize the procedure of splitting a matrix A that describes a conic consisting of two distinct lines.

1: $B := A^\triangle$;

2: Let i be the index of a nonzero diagonal entry of B;

3: $\beta = \sqrt{B_{i,i}}$;

4: $p = B_i/\beta$ where B_i is the i-th column of B;

5: $C = A + \mathcal{M}_p$;

6: Let (i, j) be the index of a non zero element $C_{i,j}$ of C;

7: g is the i-th row of C, h is the j-th column of C;

After this calculation g and h contain the coordinates of the two lines. If A describes a matrix consisting of a double line then this procedure does not apply since B will already be the zero matrix. Then one can directly split the matrix A by searching a non-zero row and a non-zero column.

11.2 The necessity of "if"-operations

Compared to all our other geometric calculations (like computing the *meet* of two lines, the *join* of two points or a *conic through five* given points) the last computation for splitting a conic is considerably different. During the computation we had to inspect a 3×3 matrix for a non-zero entry in order to extract a non-zero row (or a non-zero column). One might ask whether it is possible to perform such a computation without such an inspection of the matrix that intrinsically requires *branching* if one implements such a computation on a computer. Indeed it is not possible to to do the splitting operation for all instances without any branches. In other words: *there is no closed formula for extracting homogeneous coordinates for the two lines of a degenerate conic.* No matter which calculation is performed there will always be sporadic special cases that are not covered by the concrete formula. The reason for this is essentially of topological nature. No continuous formula can be used for performing the splitting operation without any exceptions. This is already the case for extracting the double line of a conic that consists of a double line, as the next theorem shows.

Theorem 11.1. *Let* $Deg = \{ll^T \mid l \in \mathbb{R}^3 - \{(0,0,0)^T\}\}$ *be the set of all* 3×3 *symmetric rank 1 matrices. Let* $\phi\colon Deg \to \mathbb{R}^3$ *be a continuous function that associates to each matrix* pp^T *a scalar multiple* λp *of the vector* p*. Then there is a matrix* $A \in Deg$ *for which* $\phi(A) = (0,0,0)^T$*.*

Proof. Assume that ϕ is a continuous function that associates to each matrix ll^T a scalar multiple λp of p. We consider the path $\tau : [0,\pi] \to \mathbb{R}^3$ with $\tau(t) = (\cos(t), \sin(t), 0)^T$. By our assumption the function $\phi(\tau(t))$ must be continuous on the interval $[0,\pi]$. We have $\tau(0) = (1,0,0) =: a$ and $\tau(\pi) = (-1,0,0) = -a$ and therefore $\phi(\tau(0)) = \phi(aa^T) = \phi((-a)(-a)^T) = \phi(\tau(\pi))$. By the definition of ϕ we must have $\phi(\tau(t)) = \lambda(\tau(t))$. While t moves from 0 to π the factor λ must itself behave continuously since in every sufficiently small interval at least one of the coordinates of $\tau(t)$ is constantly non-zero. However we have that $\phi(\tau(0)) = \tau(0)$ and $\phi(\tau(\pi)) = \phi(\tau(0)) = -\tau(\pi)$. This implies by the intermediate value theorem that for at least one parameter $t_0 \in [0,\pi]$ we must have $\lambda = 0$. □

The matrix ll^T that corresponds to the point p for which we have $\phi(ll^T) = (0,0,0)^T = 0 \cdot p$ does not lead to a meaningful non-degenerate evaluation of ϕ. The interpretation of this fact is a little subtle. It means that on the coordinate level there is no continuous way of extracting the double line of a conic $\mathcal{C}_A$ from a symmetric rank 1 matrix A. On the other hand the considerations of the last sections show that on the level of geometric objects there is a way to extract the coordinates of the line l from the matrix ll^T which must be necessarily continuous in the topology of our geometric objects. There is just no way of doing these computations without using branching within the calculations. This was reflected by the case that we explicitly had to search for non-zero entries in our 3×3 matrices.

In fact, the effect treated in this section is just the beginning of a long story that leads to the conclusion that elementary geometry and effects from complex function theory (like monodromy, multi-valued functions, etc.) are intimately interwoven. We will come back to these issues in the final chapters of this book.

11.3 Intersecting a conic and a line

After all this preparation work the final task of this chapter turns out to be relatively simple. We want to calculate the intersection of a conic given by a symmetric 3×3 matrix A and a line given by its homogeneous coordinates l. Clearly the task is in essence nothing else but solving a quadratic equation. However, we want to perform the operation that is as closely as possible related to the coordinate representation. For this we will use the operation of splitting a matrix that represents a degenerate conic as introduced in Section 11.1 as a basic building block. Essentially the square root needed to solve a quadratic equation will be the one required for this operation.

Our aim will be to derive a closed formula for the degenerate conic that consists of the line l as double line and whose dual consists of the two points of intersection. Such a conic is given by a pair of matrices (A, B) where A is a rank 1 symmetric matrix and describes the double line l and where B is a symmetric rank 2 matrix with $B^{\triangle} = A$ that describes the dual conic and with this the position of the two points of intersection. Splitting the matrix B yields the two points of intersection.

So, how do we derive the matrix B? We will characterize the matrix via its properties. Let p and q be the two (not necessarily distinct) intersection points. The quadratic form $m^T Bm$ must have the property that it vanishes for exactly those lines m that pass through (at least) one of the points p or q. A matrix with these properties is given by

$$B = \mathcal{M}_l^T A \mathcal{M}_l.$$

To see this we calculate the quadratic form $m^T Bm$. Using the property $\mathcal{M}_l m = l \times m$ we get

$$m^T Bm = m^T \mathcal{M}_l^T A \mathcal{M}_l m = (\mathcal{M}_l m)^T A(\mathcal{M}_l m) = (l \times m)^T A(l \times m).$$

The right hand of this chain of equation can be interpreted as follows: $l \times m$ calculates the intersection of l and m. The condition $(l \times m)^T A(l \times m)$ tests if this intersection is also on the conic $\mathcal{C}_A$. Thus as claimed $\mathcal{M}_l^T A \mathcal{M}_l$ it the desired matrix B. It is the matrix of a dual conic describing two points on l. Splitting this matrix finally gives the intersections in question.

Compared to Section 11.1 we are this time in the dual situation. We want to split a matrix of a dual conic into two *points*. For this we have to make it into an equivalent rank 1 matrix (one that defines the same conic) by adding a skew symmetric matrix. For the splitting procedure we do not have to apply the full machinery of Section 11.1. This time we are in the good situation that we already know the skew symmetric matrix up to a multiple. It must be the matrix $\mathcal{M}_l$ since l was by definition the join of the two intersection points. Thus the desired rank 1 matrix has the form

$$\mathcal{M}_l^T A \mathcal{M}_l + \alpha \mathcal{M}_l$$

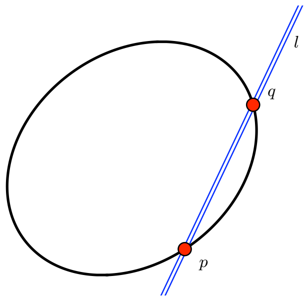

Fig. 11.1. Intersecting a conic and a line: Consider the line as a conic consisting of a double line and two points on it.

for a suitably chosen factor α. This factor must be chosen in a way that the resulting matrix has rank 1. The parameter α can be simply determined by considering a suitable 2×2 sub-matrix of the resulting matrix.

All in all the procedure of calculating the intersections of a line l and a conic given by A can be described as follows (w.l.o.g. we assume that the last coordinate entry of $l = (\lambda, \mu, \tau)^T$ is non-zero):

1: $B = \mathcal{M}_l^T A \mathcal{M}_l$;

2: $\alpha = \frac{1}{\tau}\sqrt{\begin{vmatrix} B_{1,1}, & B_{1,2} \\ B_{1,2} & B_{2,2} \end{vmatrix}}$;

3: $C = B + \alpha \mathcal{M}_l$;

4: Let (i, j) be the index of a non zero element $C_{i,j}$ of C;

5: p is the i-th row of C, q is the j-th column of C;

The choice of α in the second row ensures that the matrix C will have rank 1. The particular sign of α is irrelevant, since a sign exchange would result in interchanging the points p and q. If the entry τ of l was zero one had to take a different 2×2 matrix for determining the value of α. It is also a remarkable fact that if for some reasons one is not interested in the individual coordinates of p and q but is interested to treat them as a pair, then all necessary information is already encoded in the matrix B. Furthermore notice that for the calculation of the individual coordinates it is just necessary once to use a square-root operation. This is unavoidable since intersecting a conic and a line can be used to solve a quadratic equation.

11.4 Intersecting two conics

Now, we want to intersect two conics. For this we will use the considerations of the last sections as auxiliary primitives and will reduce the problem of intersecting two conics to the problem of intersecting a conic with a line. Intersecting a conic and a line as considered in the previous section was essentially equivalent to solving a quadratic equation. Therefore it was necessary to use at least one square-root operation. This will no longer be the case for the intersection of two conics. There the situation will be worse. Generically two conics will have *four* more or less independent intersections. This indicates that it is necessary to solve a polynomial equation of degree four for the intersection operation. However, we will present a method that only requires to solve a cubic (degree 3) equation. This results from the fact that in a certain sense the *algebraic difficulty* of solving degree three and degree four equations is essentially the same.

The idea for calculating the intersection of two conics is very simple. We assume that the the two conics $\mathcal{C}_A$ and $\mathcal{C}_B$ are represented by matrices A and B. It is helpful for the following considerations to assume that the two conics have four real intersections. However, all calculations presented here can be as well carried out over the complex numbers. All linear combinations $\lambda A + \mu B$ of the matrices represent matrices that pass through the same four points of intersection as the original matrices. In the *bundle of conics* $\{\lambda A + \mu B \mid \lambda, \mu \in \mathbb{R}\}$ we now search for suitable parameters λ and μ such that the matrix $\lambda A + \mu B$ is degenerate. After this we split the degenerate conic by the procedure described in Section 11.1. Then we just have to intersect the two resulting lines with one of the conics $\mathcal{C}_A$ or $\mathcal{C}_B$ by the procedure described in Section 11.2.

In order to get a degenerate conic of the form $\lambda A + \mu B$ we must find λ, μ such that

$$\det(\lambda A + \mu B) = 0.$$

At least one of the parameters λ or μ must be non-zero in order to get a proper conic. The problem of finding such parameters leads essentially to the problem of solving a cubic equation. To see this one can simply expand the above determinant and observe that each summand contains a factor of the form $\lambda^i \mu^{3-i}$ with $i \in \{0, 1, 2, 3\}$. Collecting all these factors leads to a polynomial equation of the form:

$$\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0.$$

We can easily calculate the parameters $\alpha, \beta, \gamma, \delta$ using the multi-linearity of the determinant function. Assume that the matrix A consists of column vectors A_1, A_2, A_3 and that the matrix B consists of column vectors B_1, B_2, B_3. Expanding $\det(\lambda A + \mu B)$ yields:

$$\begin{aligned}\det(\lambda A + \mu B) = \quad & \lambda^3\ [A_1, A_2, A_3] \\ & + \lambda^2\mu\ ([A_1, A_2, B_3] + [A_1, B_2, A_3][B_1, A_2, A_3]) \\ & + \lambda\mu^2\ ([A_1, B_2, B_3] + [B_1, A_2, B_3][B_1, B_2, A_3]) \\ & + \quad \mu^3\ [B_1, B_2, B_3].\end{aligned}$$

Thus we get

$$\begin{aligned}\alpha &= [A_1, A_2, A_3] \\ \beta &= [A_1, A_2, B_3] + [A_1, B_2, A_3] + [B_1, A_2, A_3] \\ \gamma &= [A_1, B_2, B_3] + [B_1, A_2, B_3] + [B_1, B_2, A_3] \\ \delta &= [B_1, B_2, B_3].\end{aligned}$$

If we find suitable λ, μ that solve $\alpha \cdot \lambda^3 + \beta \cdot \lambda^2\mu + \gamma \cdot \lambda\mu^2 + \delta \cdot \mu^3 = 0$ then $\lambda A + \mu B$ will represent a degenerate conic. From this degenerate conic it is easy to calculate the intersections of the original conics. Thus the problem of intersecting two conics ultimately leads to the problem of solving a cubic polynomial equation (and this is unavoidable). The story of solving cubic equations goes back to the 16th century and has a long and exciting history (this is one of *the* legends in mathematics, including human tragedies, challenges, vanity, competition. The main actors in this play were Scipione del Ferro, Antonmaria Fior, Nicolo Tartaglia, Girolamo Cradano and Lodovico Ferrari who all lived between 1465 and 1569. Unfortunately, this is a book on projective geometry and not on algebra so we do not elaborate on these stories and refer the interested reader to the books of Yaglom [**?**], the novel "Der Rechenmeister" by Joergenson [**?**] and the numerous articles on this topic on the internet).

For our purposes we will confine ourselves to a direct way for solving a cubic equation. Our procedure has three nice features compared to usual solutions of cubic equations. It works in the homogeneous setting where we ask for values of λ and μ instead of just a one variable version that does not handle "infinite" cases properly. It needs to calculate exactly one square-root and exactly one cube-root and we will not have to take care which specific roots of all complex possibilities we take. It works on the original cubic equation (most solutions presented in the literature only work for a reduced equation where $\beta = 0$.)

All calculations in our procedure have to be carried out over the complex numbers since it may happen that intermediate results are no longer real. For the computation we will need one of the third roots of unity as a constant. We abbreviate it by:

$$\omega = -\frac{1}{2} + i \cdot \sqrt{\frac{3}{4}},$$

A reasonable procedure for solving the equation

$$\alpha \cdot \lambda^3 + \beta \cdot \lambda^2\mu + \gamma \cdot \lambda\mu^2 + \delta \cdot \mu^3 = 0$$

is given by the following sequence of operations (don't ask why the parameters and formulas work, it's a long story):

1: $W = -2\beta^3 + 9\alpha\beta\gamma - 27\alpha^2\delta$;

2: $D = -\beta^2\gamma^2 + 4\alpha\gamma^3 + 4\beta^3\delta - 18\alpha\beta\gamma\delta + 27\alpha^2\delta^3$;

3: $Q = W - \alpha\sqrt{27D}$;

4: $R = \sqrt[3]{4Q}$;

5: $L = (2\beta^2 - 6\alpha\gamma, -\beta, R)^T$;

6: $M = 3\alpha(R, 1, 2)^T$;

The two vectors L and M are the key for finally finding the three solutions for the final computation of λ and μ. For this we have to compute

$$\begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \qquad \begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}.$$

The pairs (λ_1, μ_1), (λ_2, μ_2) and (λ_3, μ_3) are the three solutions of the cubic equation. In Step 4 of the above procedure we have to choose a specific cube-root. If R is one cube root the other two cube-roots are ωR and $\omega^2 R$. The reader is invited to convince himself that no matter which of these cube-roots we take we obtain the same set of solutions (in fact they are permuted). Similarly, if we change the sign of the square-root in Step 3 two of the solutions are interchanged.

Now, after collecting all these pieces it is easy to use them to give a procedure for intersecting two conics. We just have to put the pieces together. If A and B are the two matrices representing the conics we can proceed in the following way (we this time just give rough explanation of the steps instead of giving detailed formulas).

1: Calculate $\alpha, \beta, \gamma, \delta$ as described before;

2: Find a solution (λ, μ) for the cubic equation $\alpha \cdot \lambda^3 + \beta \cdot \lambda^2 \mu + \gamma \cdot \lambda \mu^2 + \delta \cdot \mu^3 = 0$;

3: Let $C = \lambda A + \mu B$;

4: Split the conic C into two lines g and h;

5: Intersect both lines g and h with the conic $\mathcal{C}_A$

All in all we obtain four intersections two for each of the two lines g and h. Figure 11.2 illustrates the process. We start with the two conics

There are a few subtleties concerning the question which intermediate results are real and which are complex. We will discuss them in the next section.

11.5 The role if complex numbers

Let us take a step back and look at what we did in the last section. At the beginning of our discussions on intersecting conics with conics we said that all

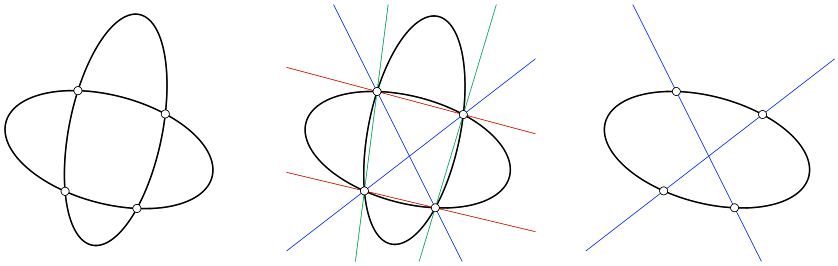

Fig. 11.2. Intersecting two conics: The original problem – the three degenerate conics – the reduced problem

calculations should be carried out over the complex numbers. In all previous Chapters we focused on *real* projective geometry. However, solving the cubic may make it inevitable to have complex numbers at least as an intermediate result (if the value of D calculated in step two of our cubic equation procedure becomes negative).

We first discuss what it means to have a *real* object in the framework of homogeneous coordinates. In all our consideration so far we agreed to identify coordinate vectors that only differ by a scalar multiple. If we work with complex coordinates we will do essentially the same however this time we will allow also complex multiples. We will call a coordinate Vector p "real" if there is a (perhaps complex) scalar s such that $s \cdot p$ has only real coordinates. In this sense the vector $(1+2i, 3+6i, -2-4i)^T$ represents a real object since we can divide by $1+2i$ and obtain the real vector $(1,3,-2)^T$. Similarly the vector $(1,i,0)$ is a proper complex vector since no which matter by which non-zero number we multiply it at least one of the entries will be complex. It is an amazing fact that we may get real objects from calculations with proper complex objects. Consider the following calculation which could be considered as the join of two proper complex points:

$$\begin{pmatrix} 1+2i \\ 3+i \\ 1-i \end{pmatrix} \times \begin{pmatrix} 1-2i \\ 3-i \\ 1+i \end{pmatrix} = \begin{pmatrix} 8i \\ -6i \\ 10i \end{pmatrix} = 2i \cdot \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}.$$

The coordinates of the result are complex but they still represent the real object $(4,3,5)^T$. The reason for this is that in this example we took the join of two complex conjugate points. More generally we get for a point $p+iq$ with real vectors p and q:

$$(p+iq) \times (p-iq) = p \times p + (iq) \times p + p \times (-iq) + (iq) \times (-iq) = 2i(q \times p).$$

The join (meet) of two conjugate complex points (lines) is a real geometric object. This perfectly fits in our geometric intuition. Imagine you intersect

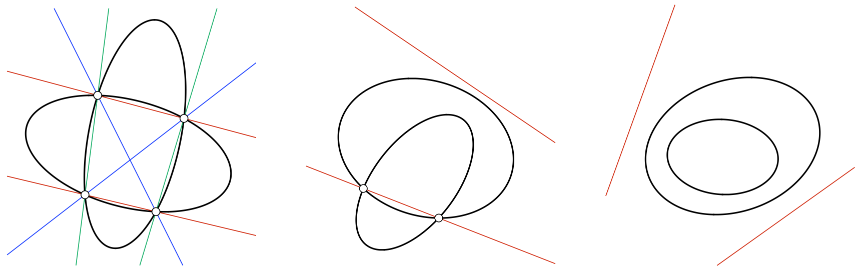

Fig. 11.3. Real degenerate conics from a pair of conics.

a line l with a conic $\mathcal{C}_A$. If the line is entirely outside the conic they do not have real intersections. As algebraic solution we get two complex conjugate points. Joining these two points we get the real line$l$ again. Generally when we deal with homogeneous coordinates we will call an object real if there is a scalar multiple that simultaneously makes all entries real. We will apply this definition to points, lines, conics, transformations and also to bundles of objects parameterized by homogeneous coordinates.

Now let us come back to the problem if intersecting conics. If we have two real conics $\mathcal{C}_A$ and $\mathcal{C}_B$ then the parameters $\alpha, \ldots, \delta$ for the real cubic will be real as well. A cubic equation in has three solutions (if necessary counted with multiplicity). Except for degenerate cases in which two or all three of these solutions coincide we will have one of the following two cases: Either all these three solutions are real or just one of the solutions is real and the others two are complex conjugates. In any case we will have at least one real solution. The term "real solution" in the homogeneous setup in which we identify scalar multiples of (λ, μ) means that there is a scalar s that makes both coordinates of $s \cdot (\lambda, \mu)$ simultaneously real. In other words, the quotient λ/μ is real. Such a real solution corresponds to the fact that the degenerate conic $C = \lambda A + \mu B$ is real again. If the conics have four points in common then we will have indeed three real solutions corresponding to the three real solutions of the cubic equation. If the two conics have only two points in common then there will be only one real solution. This solution will consists of two lines, one of these lines is the join of the intersection points the other is another line not hitting the conics in any real points, still it passes through the other two intersections of the conics which are properly complex and conjugates of each other. If the two conics do not have any real intersection then there is still a real solution to the cubic. We still get one real degenerate conic in the bundle generated by the two conics. This degenerate conic consists of two lines each of them passing through a pair of complex conjugate intersection points. The three situations are shown in Figure 11.3.

It is also very illuminating to study the case in which we pass from the "three real solutions" situation to the "one real two complex conjugates"

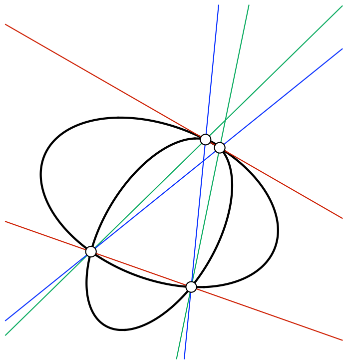

Fig. 11.4. An almost degenerate situation.

situation. In this case the cubic will a real double root and a real single root. This means that two of three real the degenerate conics in the bundle $\lambda A + \mu B$ will coincide. Geometrically this corresponds to the case in which the two conics meet tangentially at one point and have two real intersections elsewhere. Figure 11.4 shows a situation en epsilon before the tangent situation. In the picture we still have four real intersections. However, two of them approach each other tightly. It can be seen how in this case two of the degenerate conics almost coincide as well. And how one of the lines of the other conic almost becomes a tangent. In the limit case the two first degenerate conics will really coincide and the tangent line of the third conic will be really a tangent at the touching point of the two conics.

What does all this mean for our problem of calculating the intersections of two conics? If we want to restrict ourselves to real calculations whenever possible then it might be reasonable to pick the real solution of the cubic and proceed with this one. Then we finally have to intersect two real lines with a conic. Still it may happen that one or both of the lines do not intersect the conic in real points. However whenever we have real intersection points we will find them by this procedure by intersecting a real line and a real conic.

If we think of implementing such an operation in a computer program it may also be the case that the underlying math library can properly deal with complex numbers (it should anyway for solving the cubic). In such a case we do not have to explicitly pick a real solution. We can take any solution we want. If we by accident pick a complex solution then we will get a complex degenerate conic, which splits into two complex lines. However, intersecting the lines with one of the original conics will result in the correct intersections. If the correct intersections turn out to be real points then it may be just necessary to extract a common complex factor from the homogeneous coordinates.

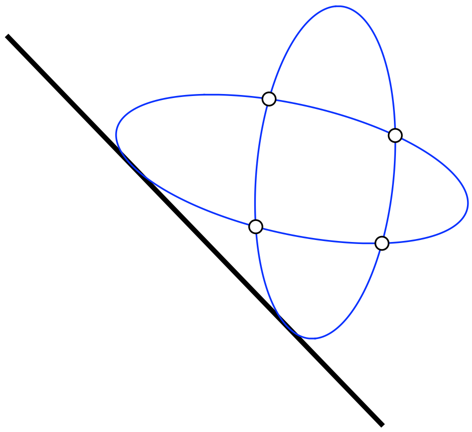

Fig. 11.5. An almost degenerate situation.

11.6 One tangent and four points

As a final example of a computation we want to derive a procedure that calculates a conic that passes through four points and at the same time is tangent to a line (see Figure 11.5 for the two possible solutions for an instance of such a problem). Based on the methods we have developed so far there are several methods to approach this construction problem. Perhaps the most straight forward one is to construct a fifth point on the conic and then calculate the conic through these five points. This fifth point can be chosen on the line itself and must be at the position where the resulting conic touches the line. Generically there are two possible positions for such a point corresponding to the two possible conics that satisfy the tangency conditions.

The crucial observation that allows us to calculate these two points is the fact that pairs of points that are generated by intersecting all possible conics through four given points with a line are the pairs of points of a projective involution on the line. For this recall that the conics through four points A, B, C, D have the form

$$\mathcal{C}_\lambda := \{p \mid (A, B; C, D)_p = \lambda\}$$

where λ may take an arbitrary value in $\mathbb{R} \cup \{\infty\}$. For each such conic there exists a pair of points on l that are as well on $\mathcal{C}_\lambda$. These pairs of points may be both real, both complex or coinciding. If the points coincide then we are exactly in the tangent situation. Let $\{p_\lambda, q_\lambda\}$ be the pair of such points on the conic $\mathcal{C}_\lambda$. Then we have:

Lemma 11.2. *There is a projective involution τ on l such that for any $\lambda \in \mathbb{R} \cup \{\infty\}$ we have $\tau(p_\lambda) = q_\lambda$.*

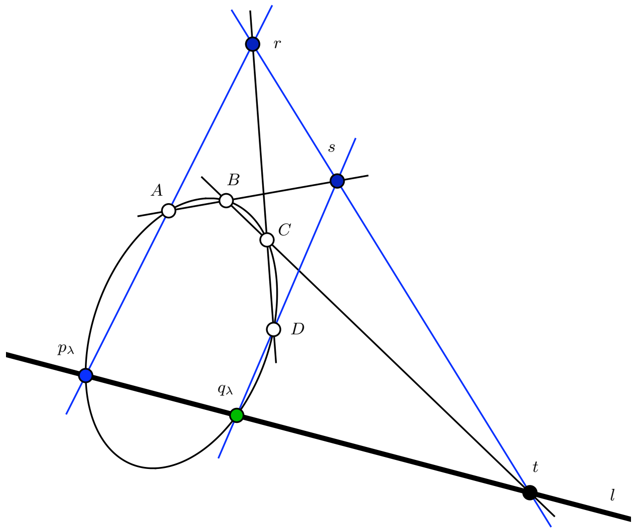

Fig. 11.6. Using Pascal's Theorem to construct the second intersection.

Proof. Instead of giving an algebraic proof we will directly use a geometric argument. With the help of Pascal's Theorem we can construct the point q_λ from A, B, C, D and p_λ without explicit knowledge of the conic $\mathcal{C}_\lambda$. Figure 11.6 shows the construction. The black and white elements of the picture only depend on A, B, C, D and p_λ. Starting with the point p_λ we have to construct first point r by intersection $\overline{p_\lambda A}$ with $\overline{CB}$, then we construct s by intersecting $\overline{rt}$ with $\overline{AB}$. Finally we derive point q_λ by intersecting $\overline{sD}$ with l. In homogeneous whole sequence of construction steps can be expressed as a sequence of cross product operations that uses the coordinates of point p exactly once. Thus if a and b are homogeneous coordinates of two arbitrary distinct points on l and we express a point p on l by $p = \alpha a + \beta b$. A sequence of operations

$$x_k \times (x_{k-1} \ldots \times (x_2 \times (x_1 \times (\alpha a + \beta b))) \ldots) =: \alpha' a + \beta' b$$

can be expressed as a single matrix multiplication

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}.$$

Thus it is a projective transformation by the fundamental theorem of projective geometry. It is clearly an involution since the same construction could be used to derive p_λ from q_λ. □

Remark 11.1. Alternatively, the proof could also be carried out in purely algebraic terms. We briefly sketch this. Let again a and b be two homogeneous coordinates of two arbitrary distinct points on l. and express a point p on l as $p = \alpha a + \beta b$. The two points on l on the conic $\mathcal{C}_\lambda$ satisfy the relation

$$[\alpha a + \beta b, A, C][\alpha a + \beta b, B, D] = \lambda[\alpha a + \beta b, A, D][\alpha a + \beta b, B, C].$$

Resolving for α and β yields the equation:

$$(\alpha, \beta)(X + \lambda Y)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$$

where X and Y are suitable symmetric 2×2 matrices in which all parameters of the first equation have been encoded. Just knowing X and Y is enough information to derive the matrix A with the property $Ap_\lambda = q_\lambda$. The matrix A can by the amazingly simple formula:

$$A = X^\triangle Y - Y^\triangle X.$$

The reader is invited to check algebraically that A is an involution and that it converts one solution of the quadratic equation into the other.

Lemma 11.2 unveils another remarkable connection between conics and quadrilateral sets. If we consider three different conics through four points A, B, C, D and consider the three point pairs that arise from intersecting these conics with line l, then these three pairs of points form a quadrilateral set. This is a direct consequence of Lemma 11.2 and Theorem 8.4 which connects projective involutions to quadrilateral sets. A corresponding picture that illustrates this fact is given in Figure 11.7. The incidence structure that is supported by the four black points and its lines is a witness that the six points on the black line form a quadrilateral set. In fact the six lines of this witness construction can themselves be considered as three degenerate conics that intersect the black line in a quadrilateral set.

Now we have collected all necessary pieces to construct the two tangent conics to l through A, B, C, D. The four points induce a projective involution τ on l which associates the pairs of points that arise by intersection with a conic through A, B, C, D. What we are looking for in order to construct a tangent conic are the two fixed points of the involution τ. Our considerations after Theorem 8.4 in Section 8.6 showed that these two fixed points x and y are simultaneously harmonic to all point pairs $(p, \tau(p))$. Thus we can reconstruct the position of these points if we know two of such point pairs by solving a quadratic equation. We can construct even three such point pairs by considering the three degenerate conics through A, B, C, D. The situation is illustrated in Figure 11.8. Considering the line l as $\mathbb{RP}^1$ and working with homogeneous coordinates on this space x and y must satisfy the equations:

$$[a_1, x][a_2, y] = -[a_1, y][a_2, x] \text{ and } [b_1, x][b_2, y] = -[b_1, y][b_2, x].$$

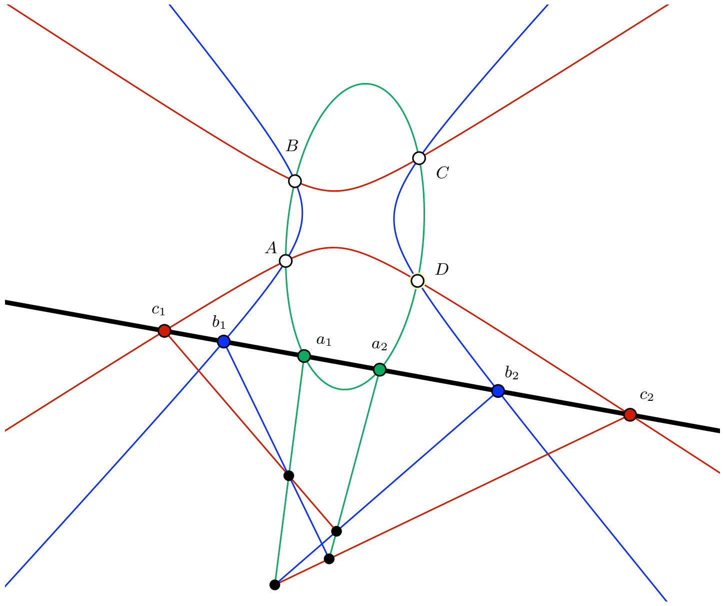

Fig. 11.7. Quadrilateral sets from bundles of conics.

These equations are solved by the two solutions

$$x = \sqrt{[a_2, b_1][a_2, b_2]}a_1 + \sqrt{[a_1, b_1][a_1, b_2]}a_2 \text{ and}$$
$$y = \sqrt{[a_2, b_1][a_2, b_2]}a_1 - \sqrt{[a_1, b_1][a_1, b_2]}a_2$$

(observe the beautiful symmetry of the solution). This solution can be derived as a variant of the Plücker's-μ technique if we try to express the solution as a linear combination $\lambda a_1 + \mu a_2$. The solution can be easily verified by plugging in the expressions for x and y in the two equations and expanding the terms.

All in all the procedure of calculating a conic through A, B, C, D tangent to l can be summarized as follows (we formulate the procedure so that the transition to the coordinates on l is only implicitly used):

1: Construct the four intersections $a_1 = \overline{AC} \wedge l$, $a_2 = \overline{BD} \wedge l$, $b_1 = \overline{AB} \wedge l$, $b_2 = \overline{CD} \wedge l$;

2: Choose an arbitrary point o not on l;

3: Let $x = \sqrt{[o, a_2, b_1][o, a_2, b_2]}a_1 + \sqrt{[o, a_1, b_1][o, a_1, b_2]}a_2$;

4: Let $y = \sqrt{[o, a_2, b_1][o, a_2, b_2]}a_1 - \sqrt{[o, a_1, b_1][o, a_1, b_2]}a_2$;

5: Return the two conics through A, B, C, D, x and through A, B, C, D, y;

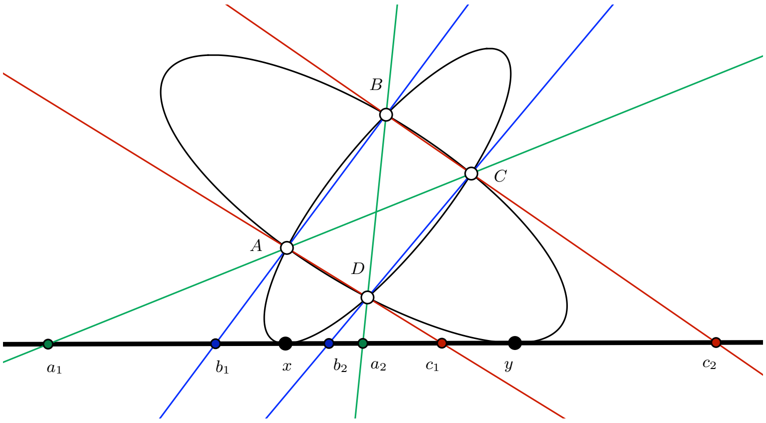

Fig. 11.8. The final construction.