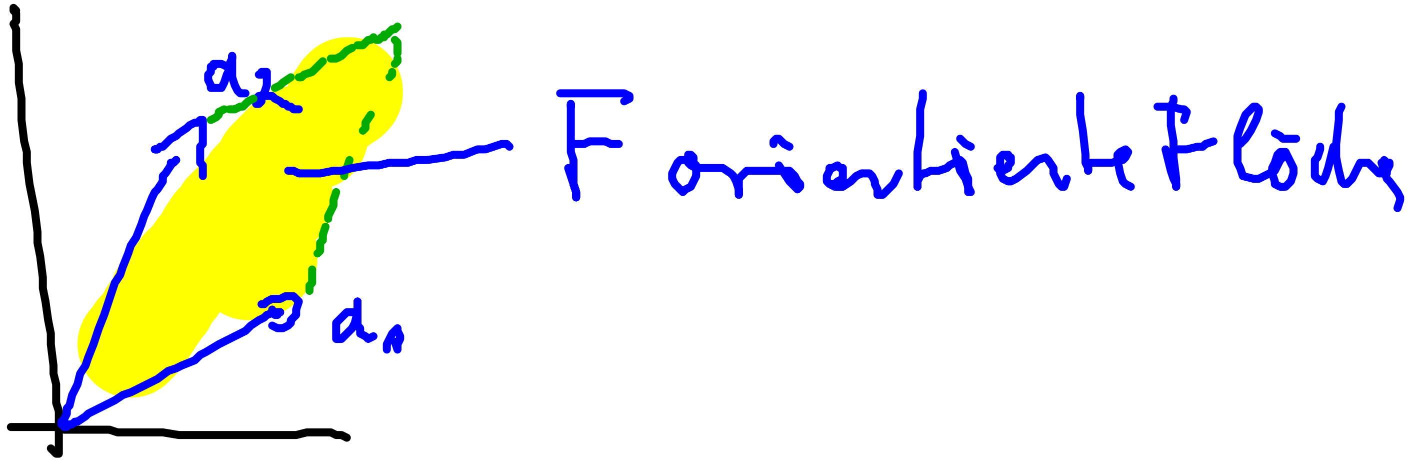
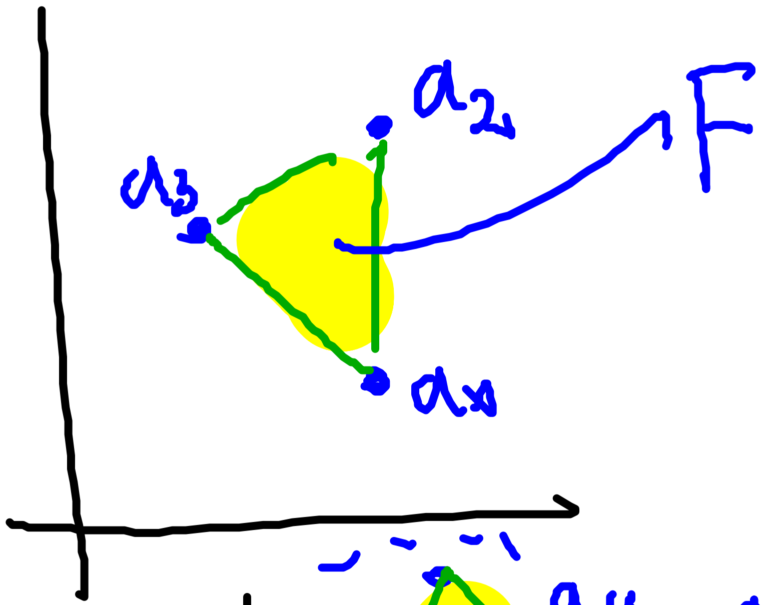
Letztes Mal: Determinanten & Fläche, Volumen

$\det(a_1, a_2)$

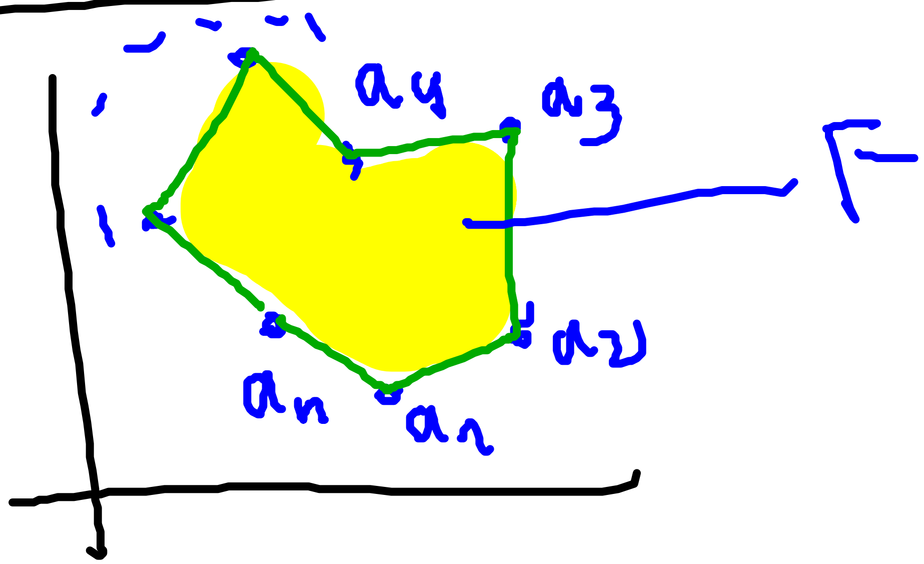


$\frac{1}{2}\det\begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \\ 1 & 1 & 1 \end{pmatrix}$

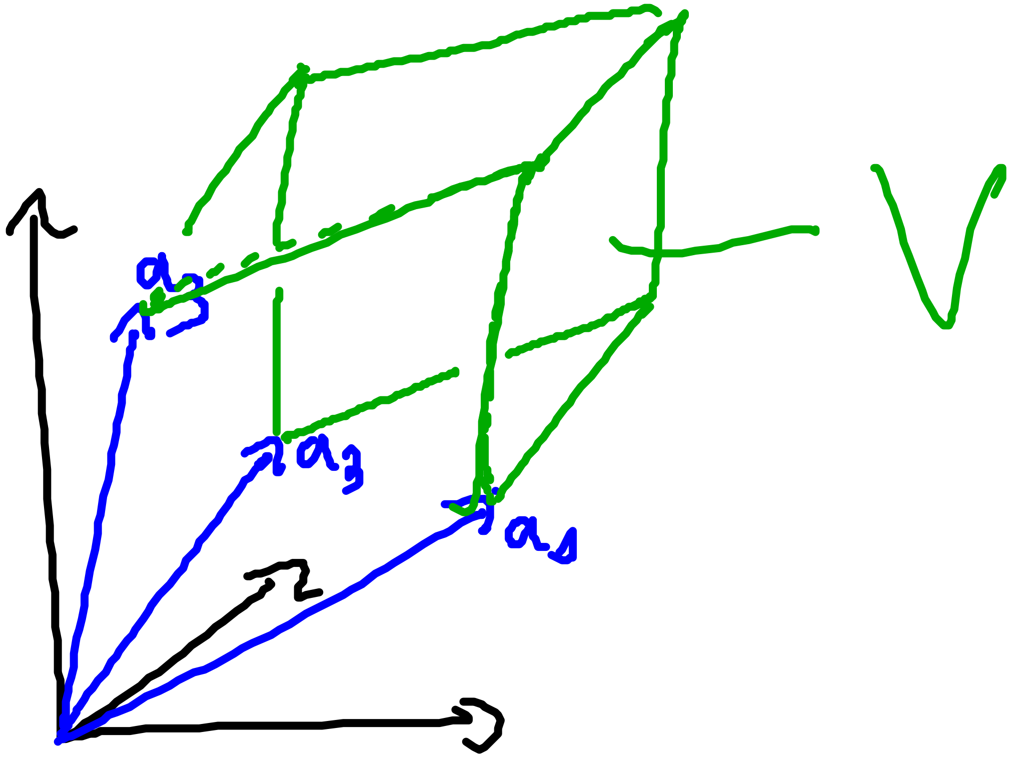


$\frac{1}{2} \cdot \sum_{i=1}^{n} \det(a_i\, a_{i+1})$

Indizes Modulo n



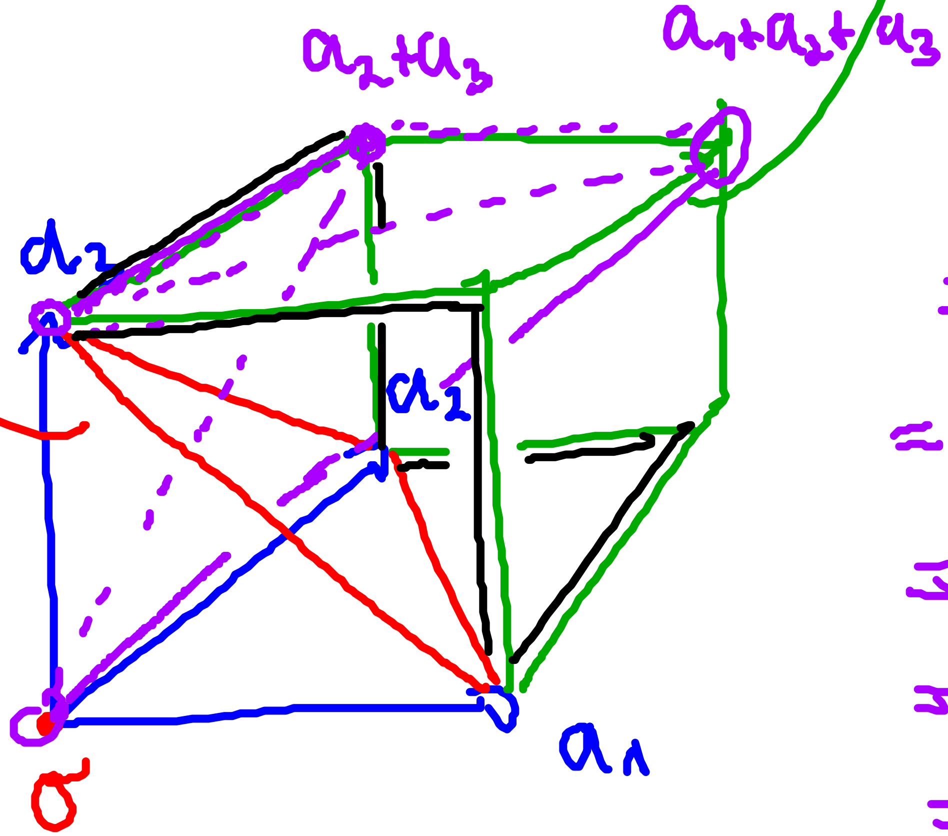
Im Raum



$\det(a_1, a_2, a_3)$

= Volumen des von $a_1\, a_2\, a_3$ aufgespannten Spates

$6 \cdot$ Tetraedervolumen $=$ Spatvolumen



$\mathrm{vol}(0, a_1, a_2, a_3)$

$= \mathrm{vol}(0, a_1+a_2+a_3, a_2+a_3, a_3)$

$= \mathrm{vol}(0, a_1+a_2+a_3, a_1+a_2, a_3)$

$\vdots$

$= \mathrm{vol}(\quad)$

$= \mathrm{vol}(\quad)$

$= \mathrm{vol}(\quad)$

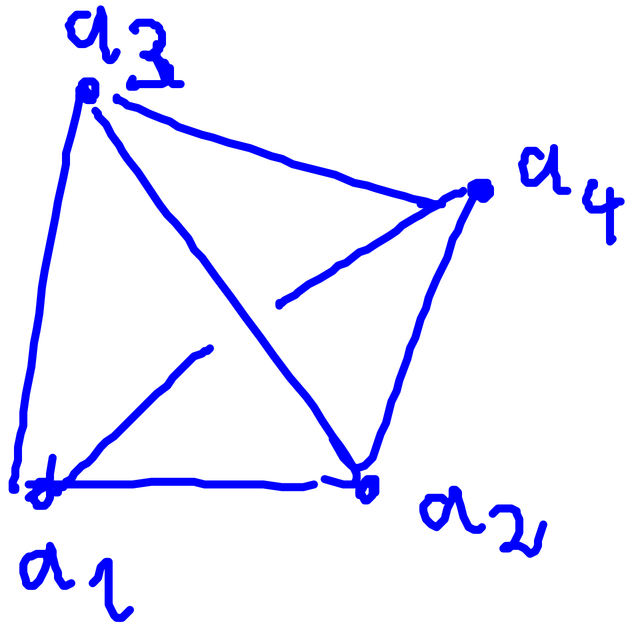
$= \mathrm{vol}(\quad)$

orientiertes Volumen

$\Downarrow$

$$\mathrm{vol}_{\triangle}(0, a_1, a_2, a_3) = \frac{1}{6} \det(a_1, a_2, a_3)$$

Volumen eines Simplex (Tetraeder)



$$\mathrm{vol}_{\triangle}(a_1, a_2, a_3, a_4)$$

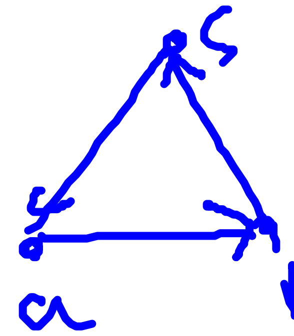
$$= \frac{1}{6} \cdot \Big(\det(a_1\ a_2\ a_3) + \det(a_2\ a_4\ a_3) + \det(a_1\ a_4\ a_2) + \det(a_1\ a_3\ a_4) \Big)$$

$$= \frac{1}{6} \cdot \det \begin{pmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_4 \\ | & | & | & | \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Randzerlegung von 3D Körpern

Sei P von endl vielen (k) Dreiecken begrenzter Körper.
Die Dreiecke seien $\Delta_i = (P_{i1}, P_{i2}, P_{i3}) \in (\mathbb{R}^3)^3$, $i = 1 \dots k$

Für ein Dreieck $(a, b, c) = \Delta$ sei



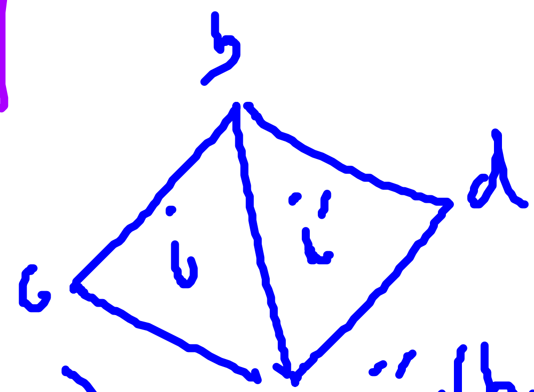
$$K(\Delta) = \{(a,b), (b,c), (c,a)\}$$

Eine Dreiecksliste $D = (\Delta_1 \dots \Delta_k)$ auf Eckpunkten $P_1 \dots P_m$ heißt orientiert falls für alle Paare (P_i, P_j)

$$|\{\Delta \mid (P_i, P_j) \in K(\Delta)\}| = |\{\Delta \mid (P_j, P_i) \in K(\Delta)\}|$$

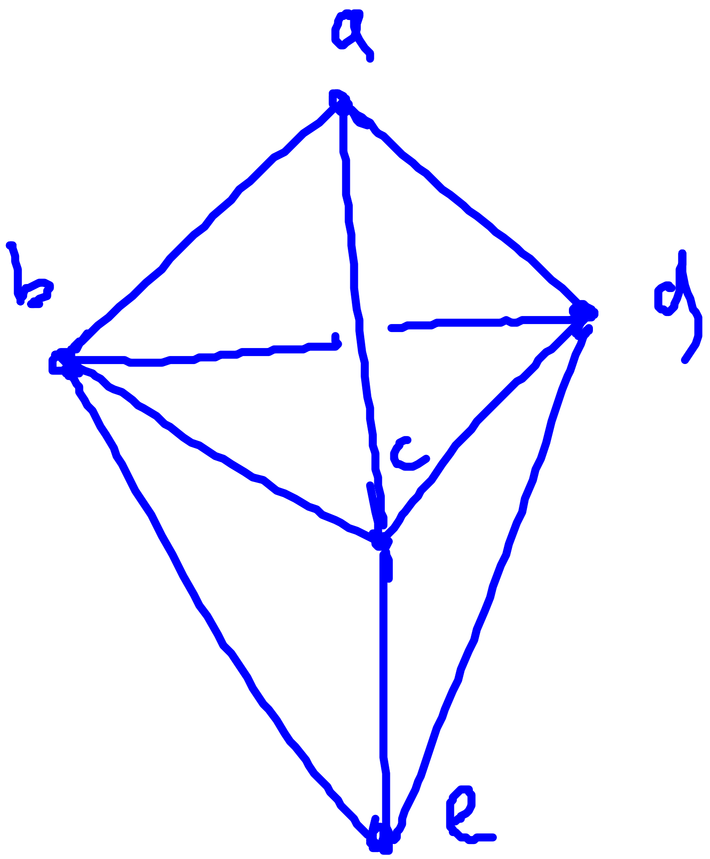
Sei D orientiert:

$$vol(P) = \frac{1}{6} \cdot \sum \det(\Delta_i)$$



$i = (a, b, c)$ $\quad$ $i' = (b, a, d)$

Beispiel Doppel Pyramide über Dreieck:

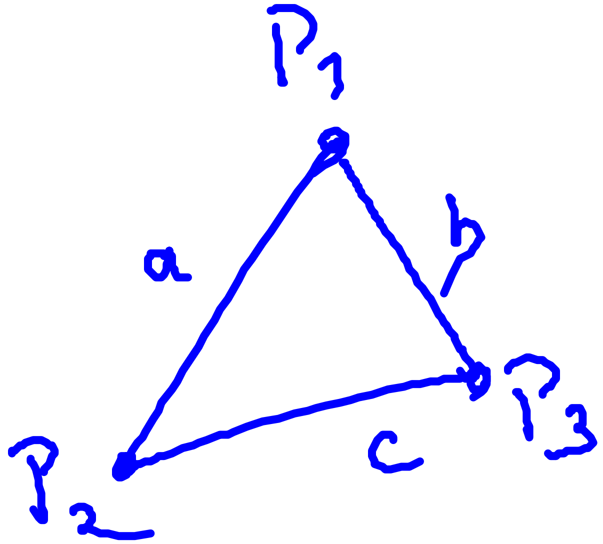


	Kanten
$\Delta_1 := (a, b, c)$	$\underline{(a\ b)}, \underline{(b\ c)}, \underline{(c\ a)}$
$\Delta_2 := (a, c, d)$	$\underline{(a\ c)}, \underline{(c\ d)}, \underline{(d\ a)}$
$\Delta_3 := (a, d, b)$	$\underline{(a\ d)}, \underline{(d\ b)}, \underline{(b\ a)}$
$\Delta_4 := (c, b, e)$	$\underline{(c\ b)}, \underline{(b\ e)}, \underline{(e\ c)}$
$\Delta_5 := (c, e, d)$	$\underline{(c\ e)}, \underline{(e\ d)}, \underline{(d\ c)}$
$\Delta_6 := (d\ e\ b)$	$\underline{(d e)}, \underline{(e b)}, \underline{(b\ d)}$

Volumen: $\frac{1}{6} \cdot \sum_{i=1}^{6} \det(\Delta_i)$

4.3 Cayley Menger Determinante

Problem: Bestimme die Fläche eines Dreiecks aus seinen Seitenlängen



$$s = \frac{a+b+c}{2}$$

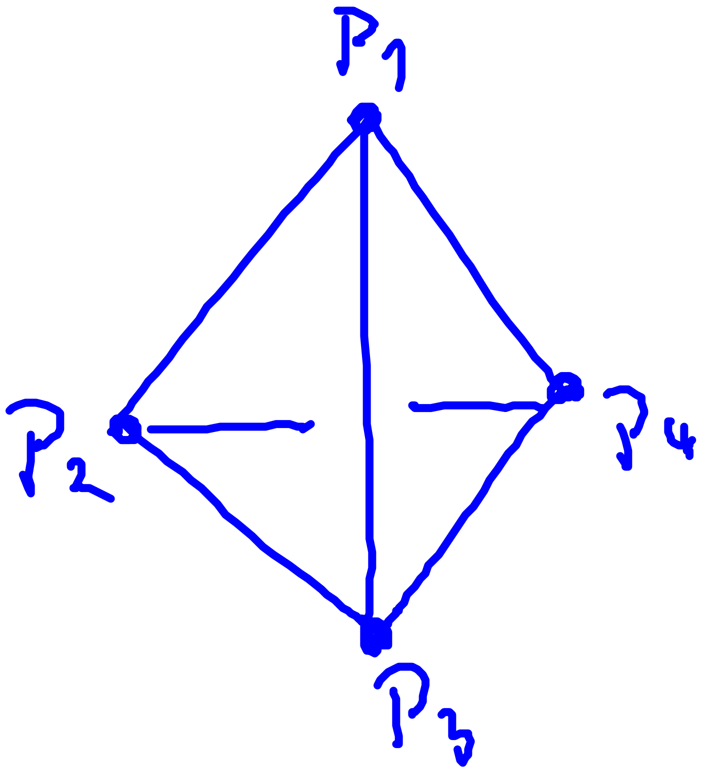
$$F_\Delta = \sqrt{s\cdot(s-a)(s-b)(s-c)}$$

Herons Formel

$$16F^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) = -\det\begin{pmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & d_{12}^2 & d_{13}^2 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

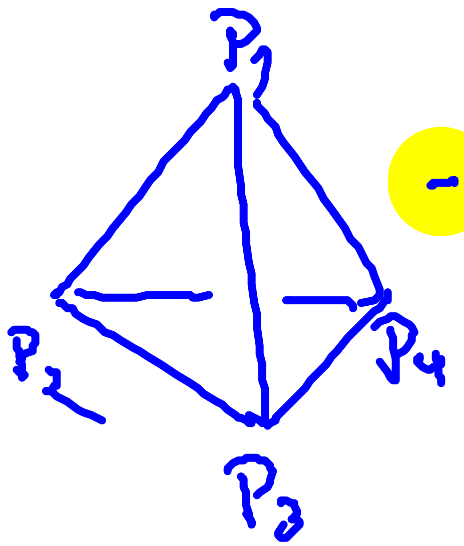
$$d_{ij}^2 = \langle P_i - P_j, P_i - P_j \rangle$$

Im $\mathbb{R}^3$ Tetraedervolumen aus Kantenlängen



$$288\,V^2 = \det\begin{pmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^3 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & d_{24}^3 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 & 1 \\ d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} \boxed{\quad d_{ij}^2 \quad} & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ 1 \;\cdots\; 1 & 0 \end{pmatrix}$$



$$-V^2 = \overbrace{\frac{1}{6}\det\begin{pmatrix} -P_1- & 1 & 0 \\ -P_2- & 1 & 0 \\ -P_3- & 1 & 0 \\ -P_4- & 1 & 0 \\ 0\;0\;0\;0 & & 1 \end{pmatrix}}^{V} \overbrace{\frac{1}{6}\cdot\det\begin{pmatrix} | & | & | & | & 0 \\ P_1 & P_2 & P_3 & P_4 & 0 \\ | & | & | & | & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}}^{-V}$$

$$= \frac{1}{36}\det\begin{pmatrix} \langle P_1,P_1\rangle & \langle P_2,P_1\rangle & \dots & & 1 \\ \langle P_1,P_2\rangle & \langle P_2,P_2\rangle & & & 1 \\ \langle P_1,P_3\rangle & & & & 1 \\ \langle P_1,P_4\rangle & \dots & & \langle P_4,P_4\rangle & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Bem $d_{ij}^2 = \langle P_i-P_j, P_i-P_j\rangle = \langle P_i,P_i\rangle + \langle P_j,P_j\rangle - 2\langle P_i,P_j\rangle$

$$\frac{1}{36}\cdot\begin{pmatrix} \frac{-d_{ij}^2+\|P_i\|^2+\|P_j\|^2}{2} & 1 \\ & 1 \\ & 1 \\ & 1 \\ 1\;1\;1\;1 & 0 \end{pmatrix} = \frac{1}{36}\cdot\frac{1}{16}\cdot\begin{pmatrix} -d_{ij}^2+\|P_i\|^2+\|P_j\|^2 & 1 \\ & 1 \\ & 1 \\ & 1 \\ 2\;2\;2\;2 & 0 \end{pmatrix}$$

$$= \frac{1}{36}\cdot\frac{1}{8}\begin{pmatrix} -d_{ij}^2 & 1 \\ & 1 \\ & 1 \\ & 1 \\ 1\;1\;1\;1 & 0 \end{pmatrix} = -\frac{1}{288}\begin{pmatrix} d_{ij}^2 & 1 \\ & \vdots \\ & 1 \\ 1 \dots 1 & 0 \end{pmatrix}$$

Allgemeine Form:

Sei S ein n-dimensionales Simplex mit Ecken $P_1, \dots, P_{n+1}$ (Position unbekannt)

Seien d_{ij} die (bekannten) Abstände zw. den Punkten

Simplexvolumen

$$\det \begin{pmatrix} 0 & d_{12}^2 & d_{13}^2 & \cdots & 1 \\ d_{12}^2 & 0 & d_{23}^2 & \cdots & \vdots \\ d_{13}^2 & d_{23}^2 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix} = (n!)^2 \cdot 2^n \cdot (-1)^{n+1} V^2$$