Aufgabe: $f'(t) = A \cdot f(t) \qquad f(0) = x_0$

Lösung: $f(t) = e^{At} \cdot x_0$

$$e^{At} = E + At + \frac{A^2 \cdot t^2}{2!} + \frac{A^3 t^3}{3!} + \ldots\ldots$$

Die obige Lösung gilt auch für nicht diagonalisierbare Matrizen.

Bsp 1: $f'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(t) \qquad f: \mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A \cdot t = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

$$(At)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot t^2 \qquad (At)^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot t^3 \qquad (At)^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot t^4$$

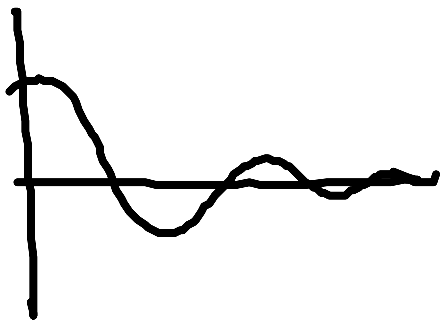
$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} -t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 0 & -t^3 \\ t^3 & 0 \end{pmatrix} + \frac{1}{4!} \begin{pmatrix} t^4 & 0 \\ 0 & t^4 \end{pmatrix} \dots\dots$$

$$e^{At} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

Bsp. 2: gedämpfte Schwingung $\quad a = v' = -x - \underbrace{r\,v}_{\text{Reibung}}$

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$



Nicht diagonalisierbar für $r = 2$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \qquad p_A(\lambda) = \lambda^2 + 2 \cdot \lambda + 1 = (\lambda + 1)^2$$

$\lambda = -1$ ist doppelter EW

$$A - \lambda E = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad v_1 = \underbrace{\boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}}_{\text{EV}} \qquad v_2 = \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$(A - \lambda E) v_2 = v_1$$

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad A = T \cdot \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} T^{-1}$$

$$e^{At} = T e^{Jt} T^{-1} = T e^{\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)t} T^{-1}$$

$$= T \underbrace{e^{\begin{pmatrix}-1 & \\ & -1\end{pmatrix}t}} \underbrace{T^{-1} \cdot T} \underbrace{e^{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}t}} T^{-1} = \begin{pmatrix}t+1 & t\\ -t & 1-t\end{pmatrix} e^{-t}$$

$$T \left(e^{-t} \cdot E \cdot \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix}\right)\right) T^{-1}$$

$$\boxed{x(t) = \alpha e^{-t} + \beta t e^{-t}}$$

4 Mehr über Determinanten

4.1. Was bisher geschah: (Im Folgenden alle Matrizen $n \times n$)

- Axiomatische Definition

  $\det : \mathbb{K}^{n \times n} \to \mathbb{K}$

  (i) Linear in jeder Spalte
  (ii) Antikommutativ
  (iii) $\det(E) = 1$

- Die drei Eigenschaften legen $\det(\ldots)$ eindeutig fest.

- Wichtige Formeln:

$$\det(A) = \det(A^T)$$
$$\det(A \cdot B) = \det(A) \cdot \det(B)$$
$$\det(A^{-1}) = 1/\det(A)$$

- Determinanten und lineare Abhängigkeit:

$\det(A) = 0 \iff$ Spalten linear abhängig

$\det(A) = 0 \iff \operatorname{Kern}(A) \neq \{0\}$

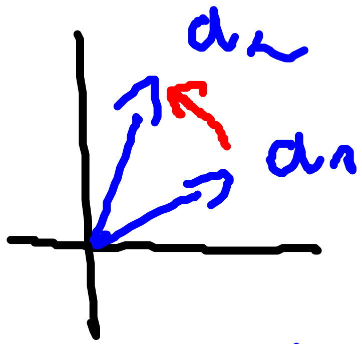
- geometrische Deutung

$\det(A) = 0 \Rightarrow$ Spaltenvektoren in gemeinsamer Hyperebene

4.2. Volumen + Orientierung:

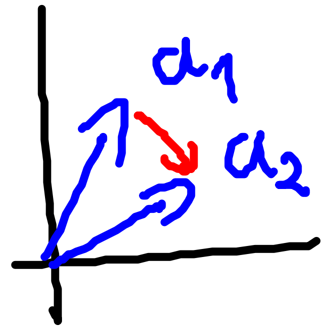
Orientierung:

$\mathbb{R}^2$: $\sigma(a_1\ a_2) = \operatorname{sign}(\det(a_1\ a_2))$



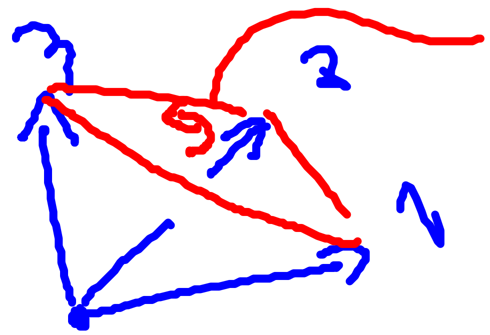
$\sigma = +$

$\sigma = 0$



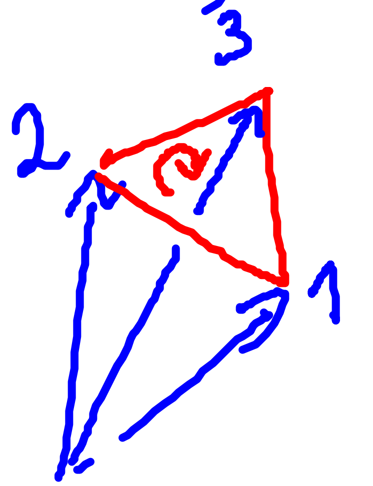
$\sigma = -$

$\mathbb{R}^3$: $\sigma(a_1, a_2, a_3) = \operatorname{sign}(\det(a_1\ a_2\ a_3))$



gegen den Uhrzeigersinn

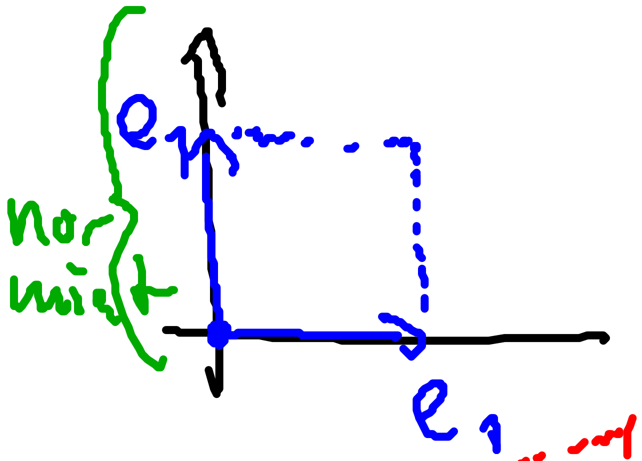
$\sigma = 1$



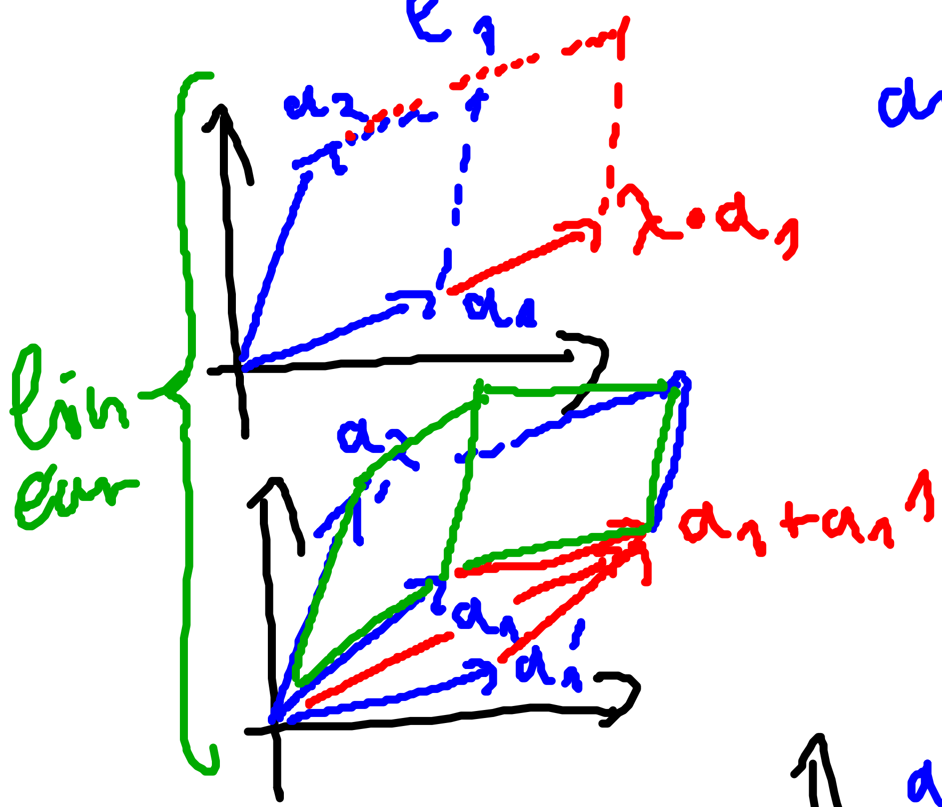
$\sigma = -$

„Rechte Hand Regel“

orientierte Fläche von Parallelogramme

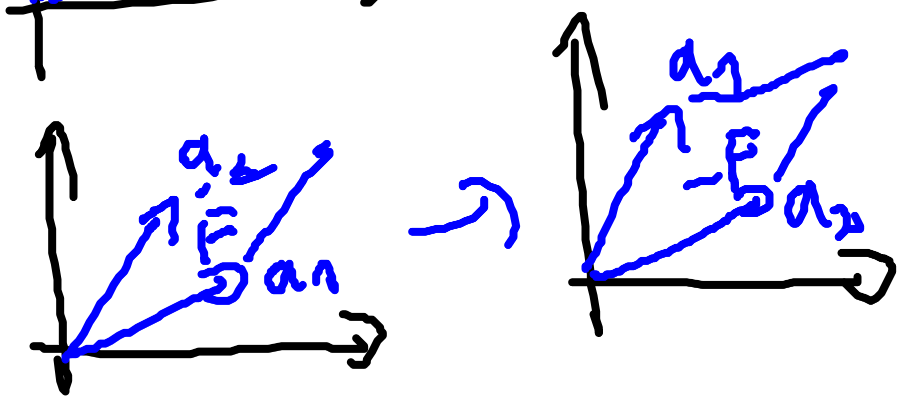


$$\text{area}_{▱}(e_1, e_2) = 1$$



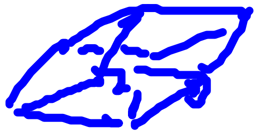
$$\text{area}_{▱}(\lambda a_1, a_2) = \lambda \, \text{area}_{▱}(a_1, a_2)$$

$$\text{area}_{▱}(a_1 + a_1', a_2) = \underline{\text{area}_{▱}(a_1, a_2)} + \underline{\text{area}_{▱}}(a_1', a_2)$$



$$\text{area}_{▱}(a_1\, a_2) = -\text{area}_{▱}(a_2\, a_1)$$

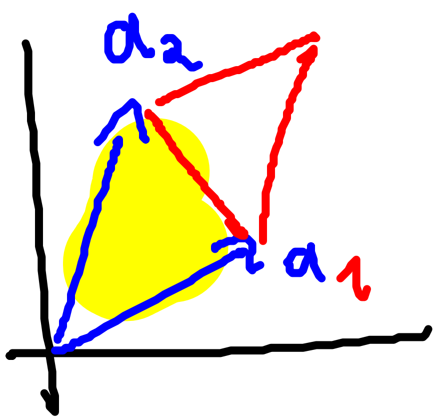
$$\boxed{\Rightarrow \text{area}_{▱} = \det}$$

$\mathbb{R}^3$ $\quad \mathrm{vol}$  $(a_1\ a_2\ a_3) = \det(a_1\ a_2\ a_3)$

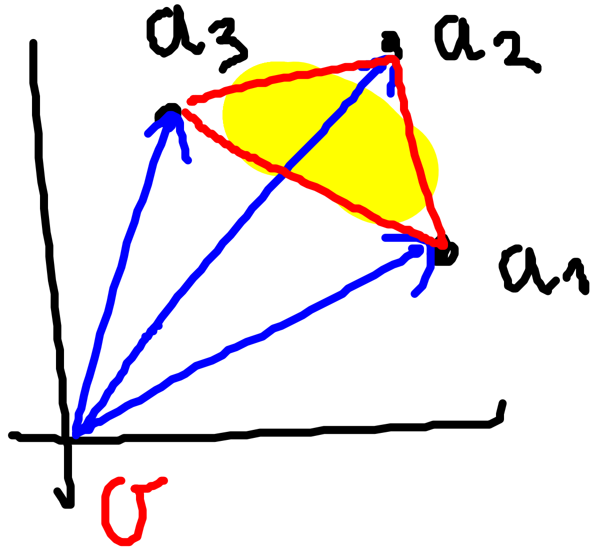
Allgemein

$$\mathrm{vol}_{\text{parallelotop}}(a_1, \ldots, a_n) = \det(a_1, \ldots, a_n)$$

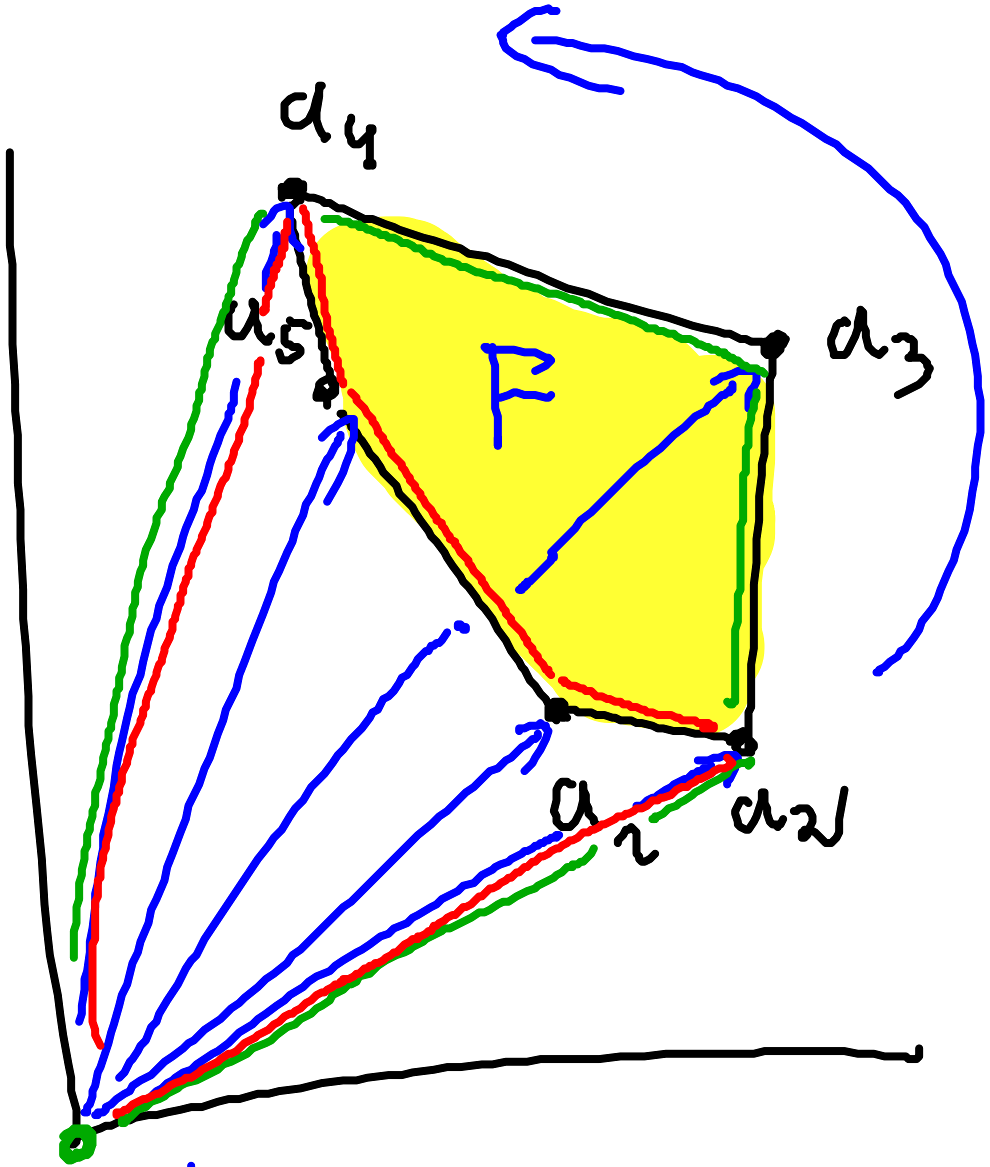
Fläche von Dreiecken



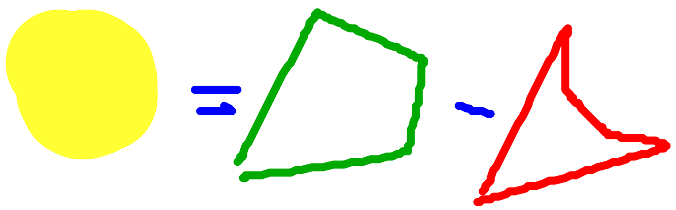
$$\text{area}_\Delta(0, a_1, a_2) = \frac{1}{2}\det(a_1, a_2)$$



$$\text{area}_\Delta(0, a_1, a_2) + \text{area}_\Delta(0, a_2\ a_3)$$
$$- \text{area}\,\Delta(0, a_1\ a_3)$$
$$= \frac{1}{2}\cdot\left(\det(a_1, a_2) + \det(a_2\ a_3) - \det(a_1\ a_3)\right)$$
$$= \det\begin{pmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \\ 1 & 1 & 1 \end{pmatrix}\cdot\frac{1}{2}$$

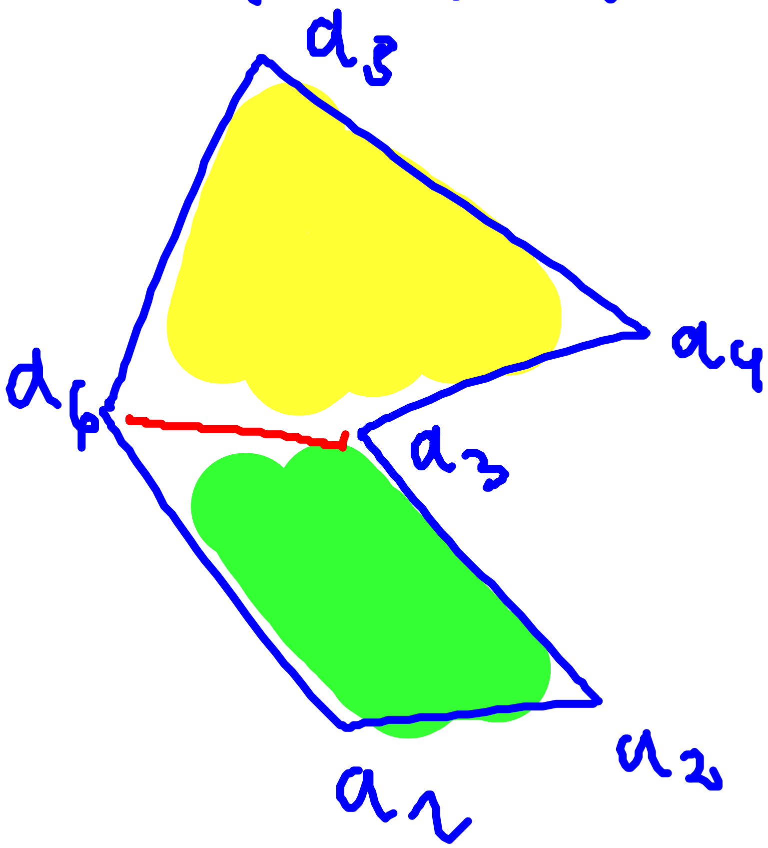


$$
\begin{aligned}
2 \cdot \text{area}(F) = \; & \det(a_1\; a_2) \\
+ \; & \det(a_2\; a_3) \\
+ \; & \det(a_3\; a_4) \\
+ \; & \det(a_4\; a_5) \\
+ \; & \det(a_5\; a_1)
\end{aligned}
$$

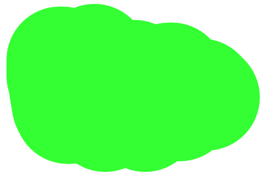


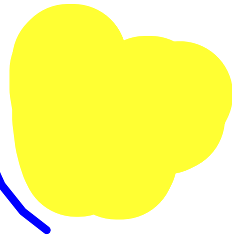
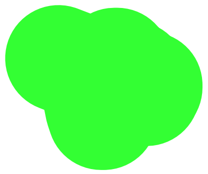
Nummerierung der Reihe nach gegen Uhrzeigersinn

Was Passiert bei nicht konvexen Polygonen



$2 \cdot$ $= \det(a_3\ a_4) + \det(a_4\ a_5) + \det(a_5\ a_6) + \underline{\det(a_6\ a_3)}$

$2 \cdot$  $= \det(a_1\ a_2) + \det(a_2\ a_3) + \underline{\det(a_3\ a_6)} + \det(a_6\ a_1)$

$2 \cdot \Big($  $+$  $\Big) = \det(a_1\ a_2) + \det(a_2\ a_3) \ldots\ldots + \det(a_5, a_6) + \det(a_6\ a_1)$

Selbstüberschneidungen: