Letztes Mal: Wie macht man aus $\underbrace{(\mathbb{R}^2,+)}_{\text{gruppe}}$ einen Körper

$\Rightarrow$ Zahlen der Form $a+ib$; $a,b\in\mathbb{R}$

a: Realteil, b: Imaginärteil

$\mathbb{C}=\mathbb{R}+i\mathbb{R}\approx\mathbb{R}^2$

komplexe Zahlen

zusätzliche Regel $i^2=-1$

$(a_1+ib_1)+(a_2+ib_2)$

$=(a_1+a_2)+i(b_1+b_2)$

$(a_1+ib_1)\cdot(a_2+ib_2)$

$=(a_1\cdot a_2-b_1b_2)+i(a_1b_2+a_2b_1)$

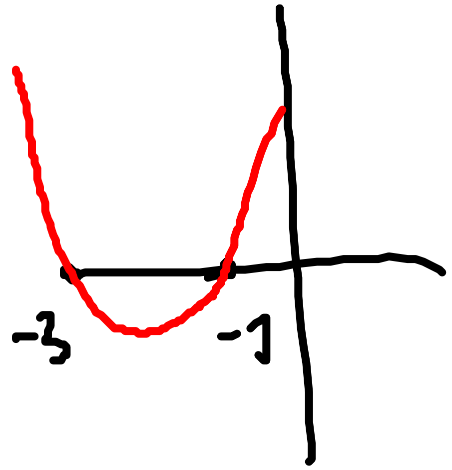
Eine Motivation: Nullstellen von quadratischen Polynomen $x^2+px+q=0$

$$x_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4}-q}$$

Bsp: Suche x mit $x^2+4x+3=0$

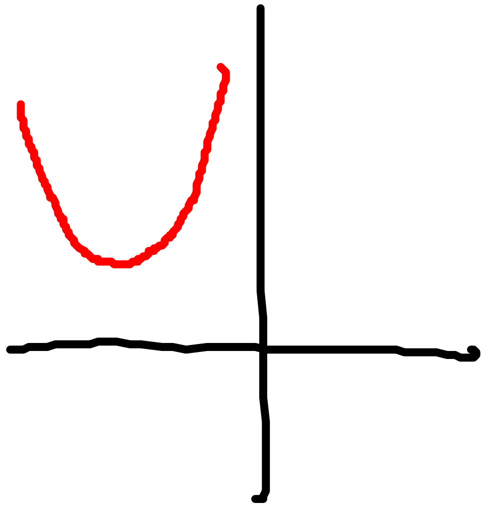
$$x_{1/2} = -2 \pm \sqrt{4-3} = -2 \pm 1$$

$$\Rightarrow x_1 = -1$$
$$x_2 = -3$$



Suche x mit $x^2+4x+5=0$

$$x_{1/2} = -2 \pm \sqrt{4-5} = -2 \pm \sqrt{-1}$$



$$x_1 = -2+i$$
$$x_2 = -2-i$$

Probe! $x = -2+i$

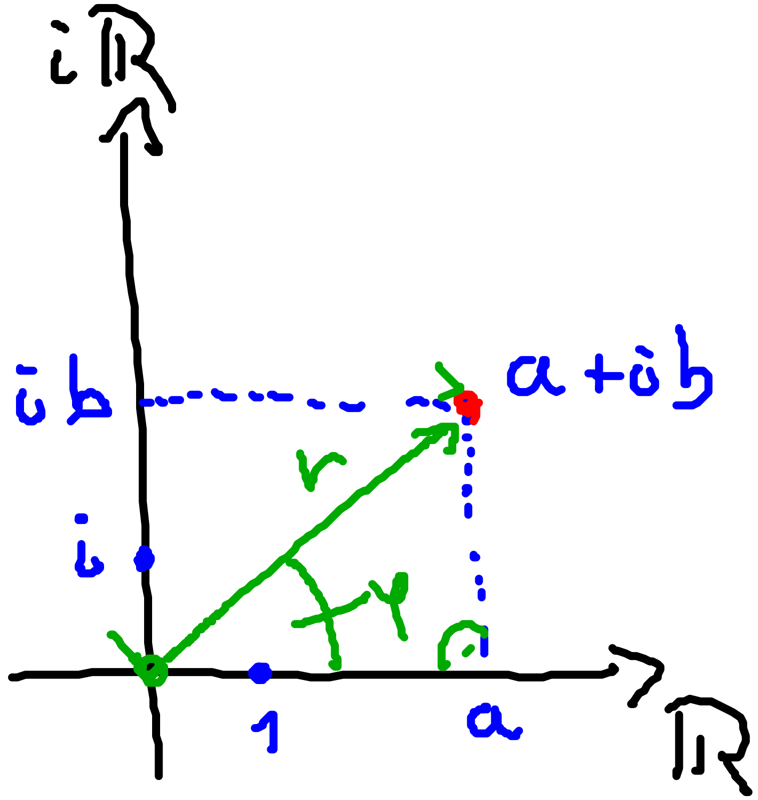
$$x^2 + 4x + 5 = 0$$

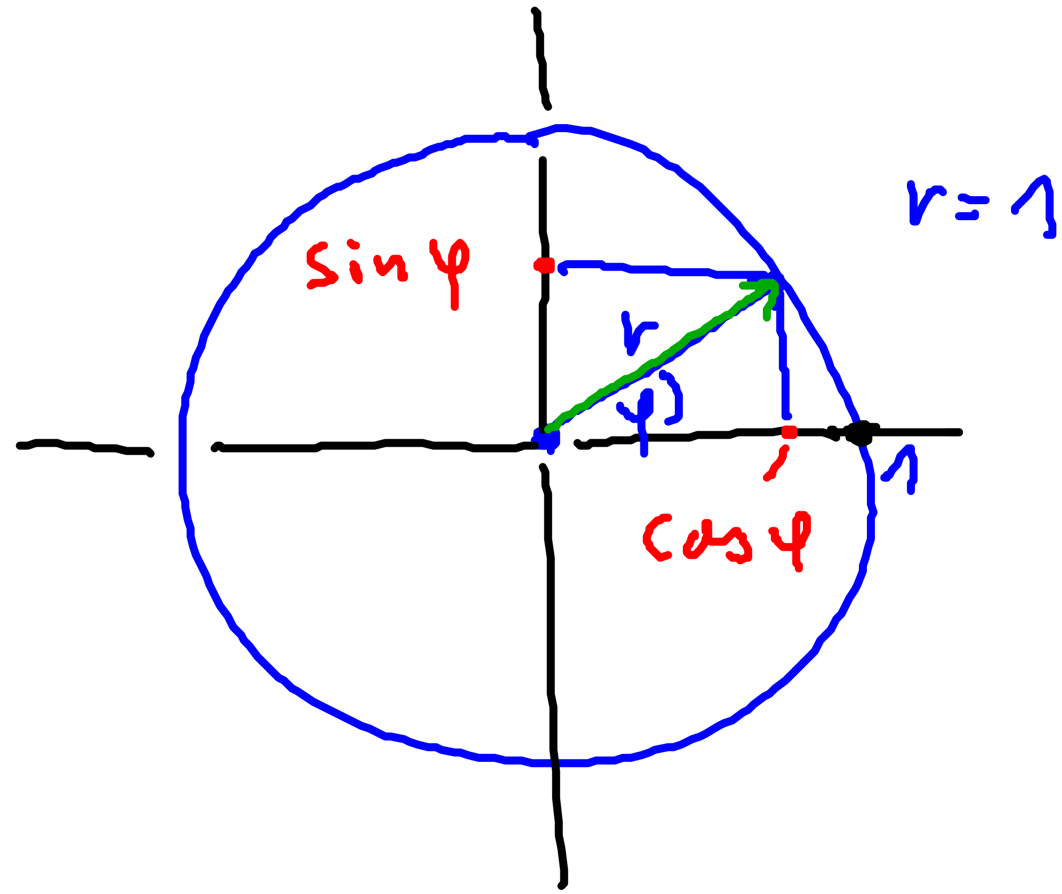
$$(-2+i)(-2+i) + 4(-2+i) + 5$$

$$= 4 - 1 - 2i - 2i - 8 + 4i + 5$$

$$= 0 + i \cdot 0 = 0$$

Geometrische Interpretation von $\mathbb{C}$





$$r = \sqrt{a^2+b^2}$$

$$\varphi = \arctan(b/a)$$

$$2{,}71828\ldots = \sum_{i=0}^{n} \frac{1}{n!}$$

$$a+ib = r\cdot(\cos\varphi + i\sin\varphi) = r\cdot e^{i\varphi}$$

Sonderfall

$$e^{i\pi}+1=0$$

MathTopModel

Aus der Analysis: Potenzreihen

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

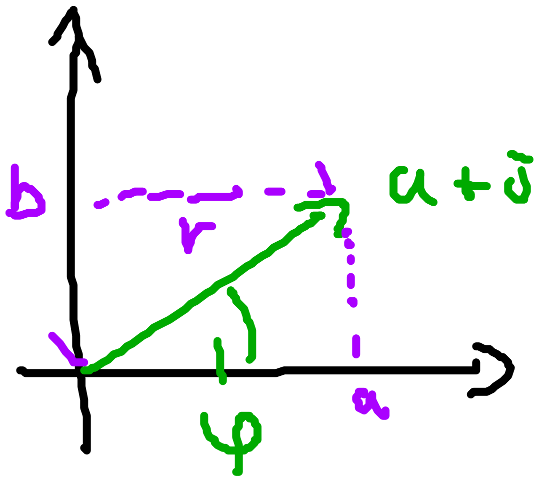
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots$$

$$e^{ix} = \boxed{1} + \boxed{ix} - \boxed{\frac{x^2}{2!}} - \boxed{i\frac{x^3}{3!}} + \boxed{\frac{x^4}{4!}} + \boxed{i\frac{x^5}{5!}} - \boxed{\frac{x^6}{6!}} - \boxed{i\frac{x^7}{7!}} + \boxed{\frac{x^8}{8!}}$$

$$i\sin(x) = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} \dots$$

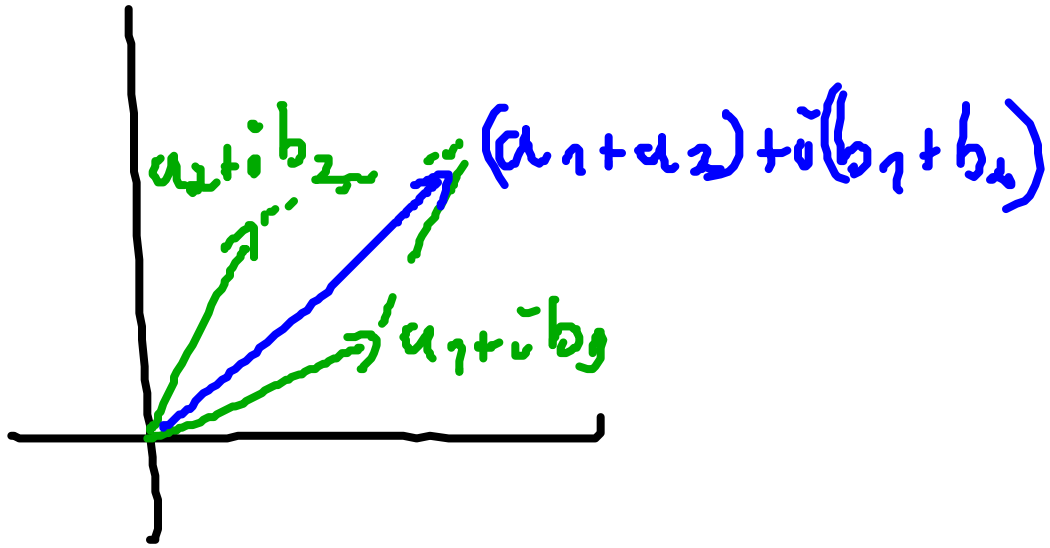
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \dots$$

$$e^{ix} = \cos(x) + i\sin(x)$$



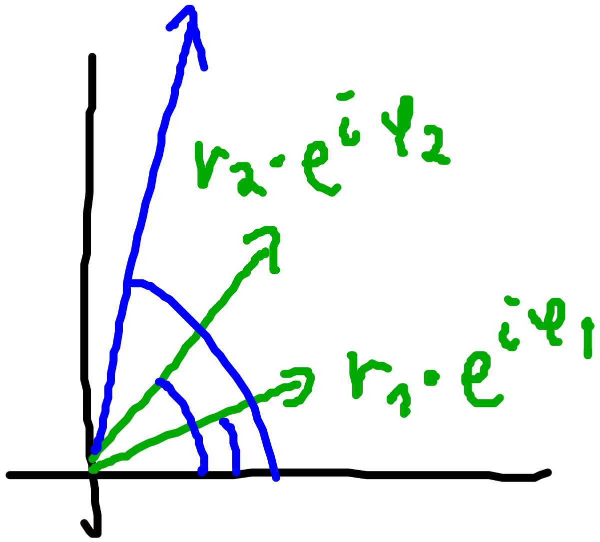
$a+ib = r\cdot e^{i\varphi}$

Addition



Addition ~ Verschiebung

Multiplikation



$$r_1 e^{i\varphi_1}\cdot r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)}$$

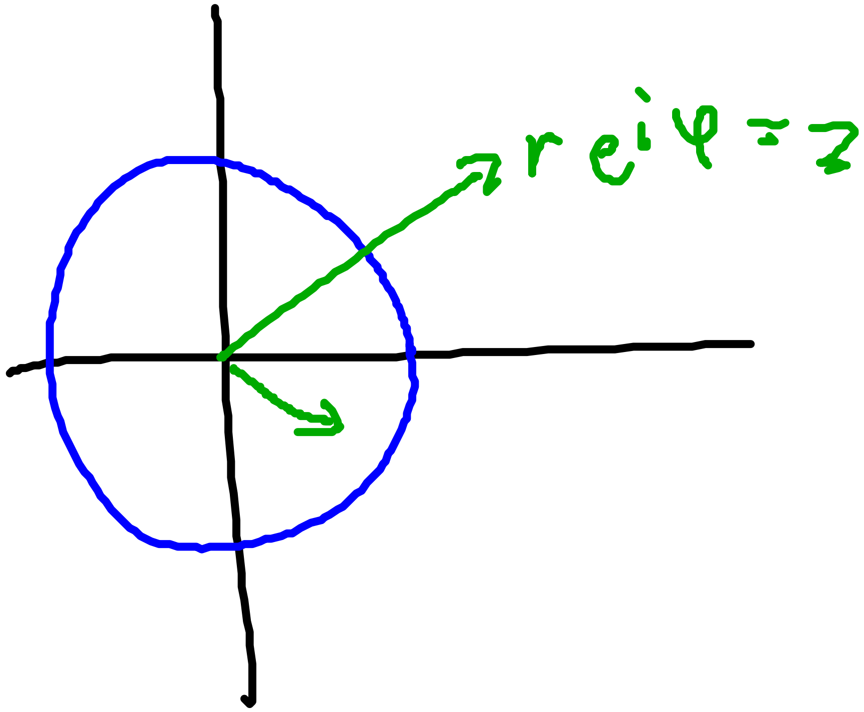
Längen multiplizieren sich

Winkel addieren sich

~ Drehstreckung

Multiplikatives Inverses

$$z = r \cdot e^{i\varphi}$$



$$\frac{1}{z} = \frac{1}{r} \cdot e^{-i\varphi}$$

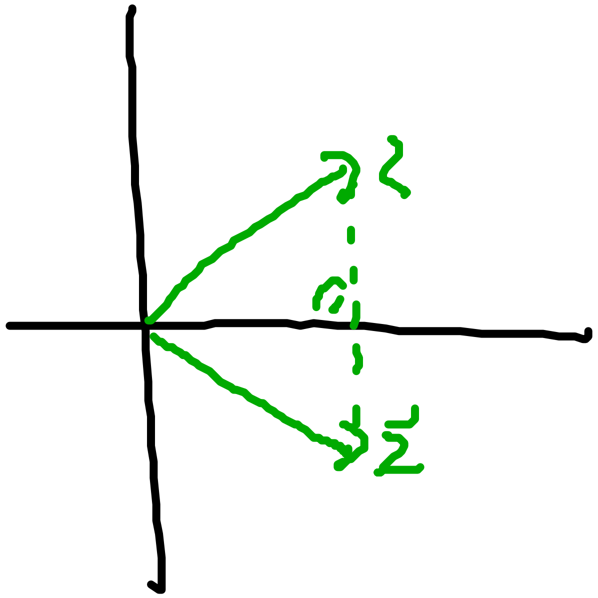
Probe: $(r \cdot e^{i\varphi}) \cdot \left(\frac{1}{r} e^{-i\varphi}\right)$

$$= \frac{r}{r} \cdot e^{i(\varphi - \varphi)} = 1 \cdot e^{0} = 1$$

komplexes Konjugieren

$$z = a + ib = r e^{i\varphi}$$

$$\bar{z} = a - ib = r e^{-i\varphi}$$



Wichtige Anwendung

$$|z| := r = \sqrt{z \cdot \bar{z}}$$

$$= \sqrt{(a+ib)(a-ib)}$$

$$= \sqrt{a^2 + b^2}$$

Potenzieren einer komplexen Zahl $z = r \cdot e^{i\varphi}$

$z^2 = r^2 \cdot (e^{i\varphi})^2 = r^2 \cdot e^{i2\varphi}$ ⟸ Doppelte Winkel von z

$z^3 = r^3 \cdot (e^{i\varphi})^3 = r^3 \cdot e^{i3\varphi}$ ⟸ Dreifachen Winkel

$z^4 = r^4 \cdot (e^{i\varphi})^4 = r^4 \cdot e^{i4\varphi}$

In der reellen Analysis üblich

$\sqrt{\ldots} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$

$\sqrt[k]{\ldots} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$

Bsp $\sqrt[3]{8} = 2$

Hier anderer Ansatz

$\sqrt{x}$ ≈ eine Zahl die mit sich selbst multipliziert x ergibt.

Lösung der Gleichung $\omega^2 = x$

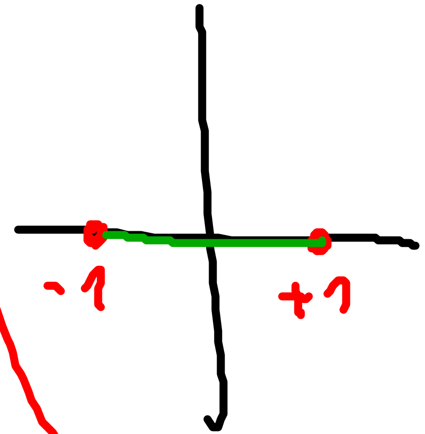
Bsp $x = 1$, $\omega_1 = 1$, $\omega_2 = -1$

Wir sind an allen Lösungen von $\omega^k = z$

Zunächst: Einheitswurzeln: Lösungen von $\omega^k = 1$

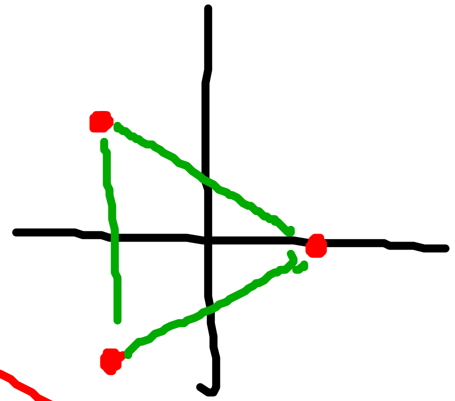
gesucht alle $\omega \in \mathbb{C}$ mit $\omega^k = 1$

$k=2$: $\omega_0 = 1$, $\omega_1 = -1 = e^{i\pi}$



$k=3$: $\omega_0 = 1$, $\omega_1 = e^{i\frac{2\pi}{3}}$, $\omega_2 = e^{-i\frac{2\pi}{3}}$

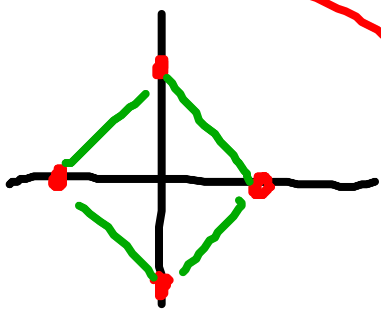
$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$



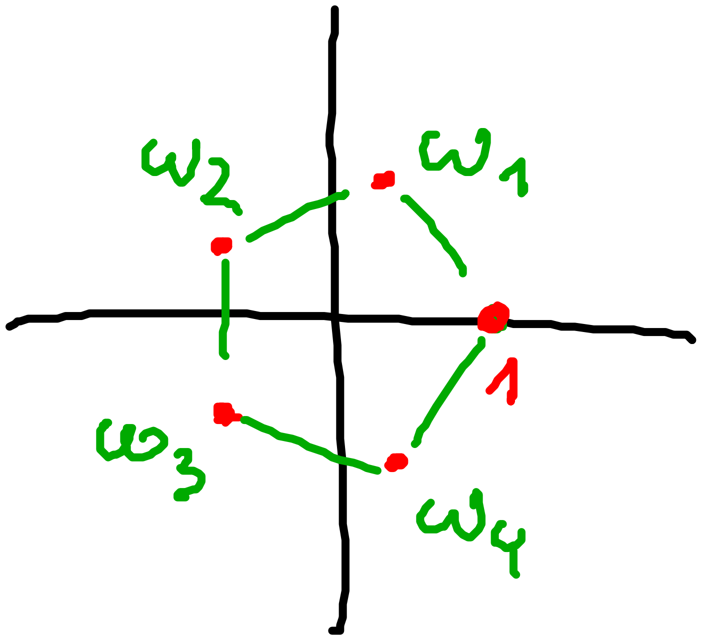
$k=4$:

$\omega_0 = 1$, $\omega_1 = i$, $\omega_2 = -1$, $\omega_3 = -i$

$\ = e^{i\frac{\pi}{2}}$, $= e^{-i\pi}$, $= e^{i\frac{3\pi}{2}}$



$k = 5$ .....



Allgemein: Lösungen von $\omega^k = 1$

$$\omega_n = e^{i n \frac{2\pi}{k}} \qquad n \in \{0, \dots, k-1\}$$

Probe: $(\omega_n)^k = \left(e^{i n \frac{2\pi}{k}}\right)^k = e^{i n \frac{2\pi \cdot k}{k}} = e^{i n 2\pi} = \left(e^{i 2\pi}\right)^n$

$= 1^n = 1$