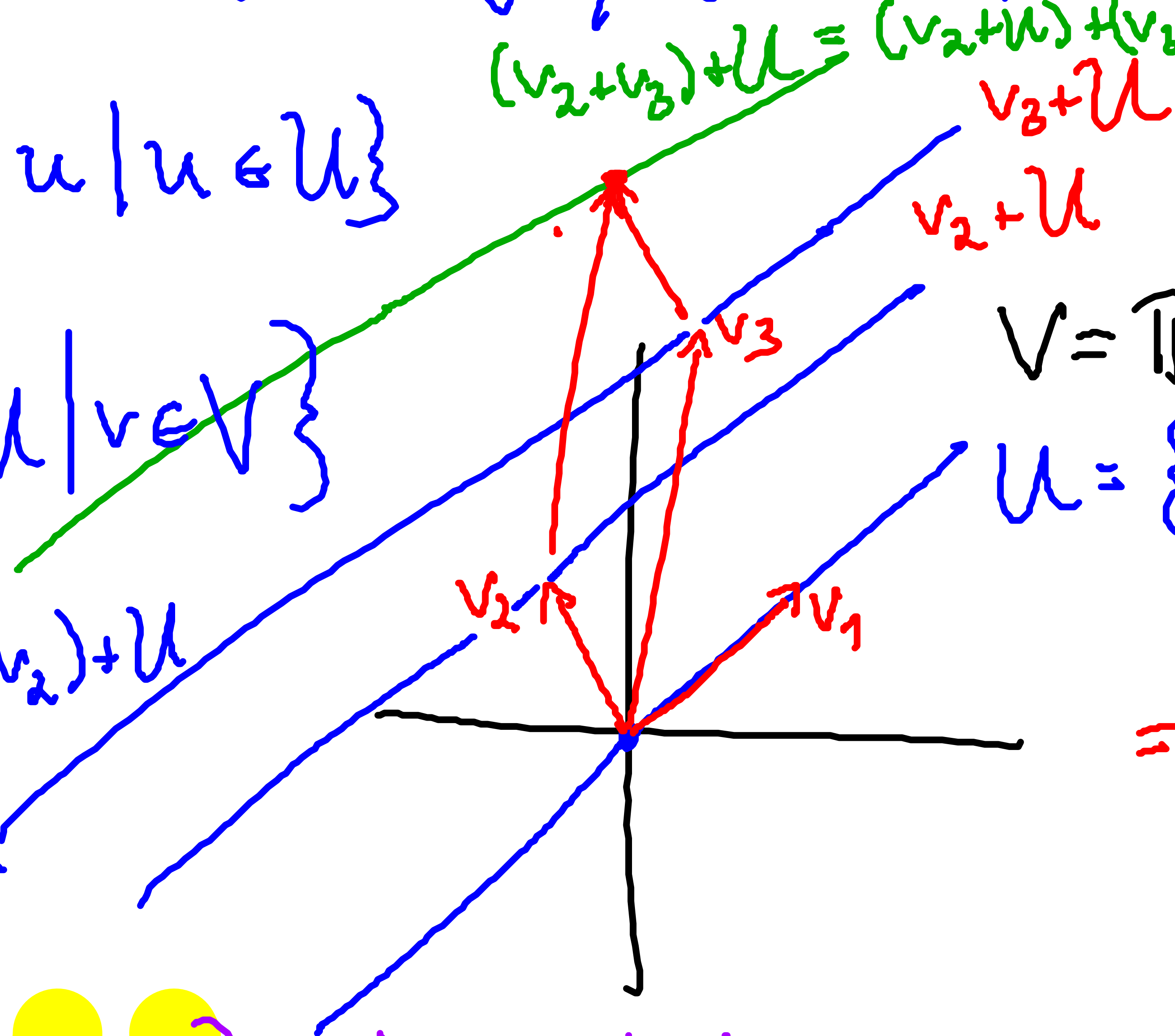
Letztes Mal: Quotientenräume / Nebenklassen

Gegeben Vektorraum V, Untervektorraum U

$$v+U := \{v+u \mid u \in U\}$$

$$V/U := \{v+U \mid v \in V\}$$



$V = \mathbb{R}^2$

$U = \{\binom{x}{y} \in \mathbb{R}^2 \mid x-y=0\}$

$$(v_1+U)+(v_2+U) = (v_1+v_2)+U$$

$$\lambda \cdot (v+U) = \lambda v + U$$

$(V/U, K, +_K, \cdot_K, +, \cdot)$ ist ein Vektorraum

Isomorphiesatz: $f: V \to W$ lin. Abb.

$$V/\mathrm{Kern}(f) \cong \mathrm{Bild}(f)$$

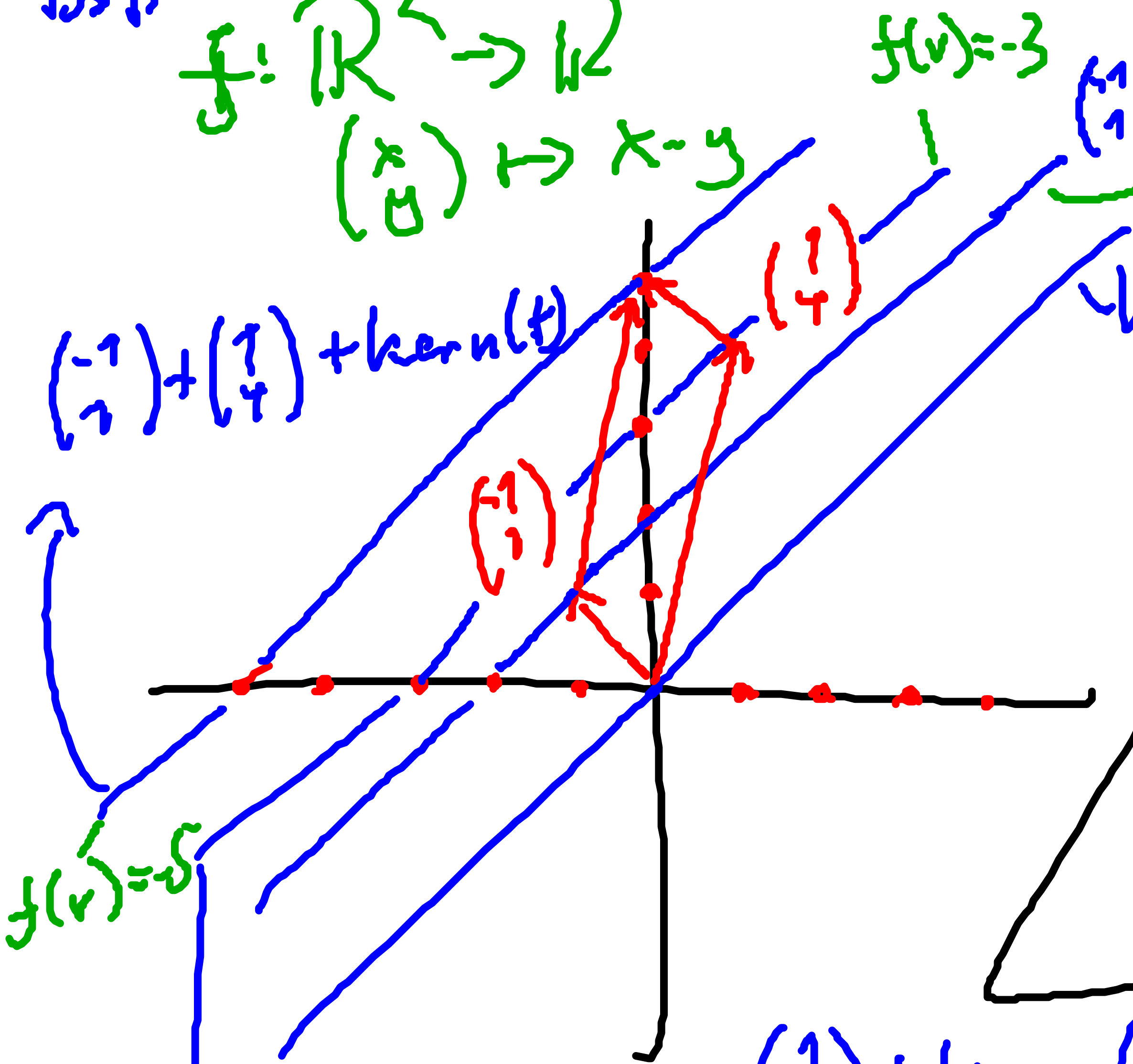
$$V/\mathrm{Kern}(f) = \{v + \mathrm{Kern}(f) \mid v \in V\}$$

$$\mathrm{Bild}(f) = \{f(v) \mid v \in V\}$$

Bsp

$f: \mathbb{R}^2 \to \mathbb{R}$

$\binom{x}{y} \mapsto x - y$



$$\binom{-1}{1} + \mathrm{Kern}(f) = \left\{ \binom{x}{y} \;\middle|\; f\binom{x}{y} = f\binom{-1}{1} \right\}$$

$$\Phi: V/\mathrm{Kern}(f) \to \mathrm{Bild}(f)$$
$$v + \mathrm{Kern}(f) \mapsto f(v)$$

- wohldefiniert
- bijektiv
- $\Phi(a+b) = \Phi(a) + \Phi(b)$
- $\Phi(\lambda \cdot a) = \lambda \cdot \Phi(a)$

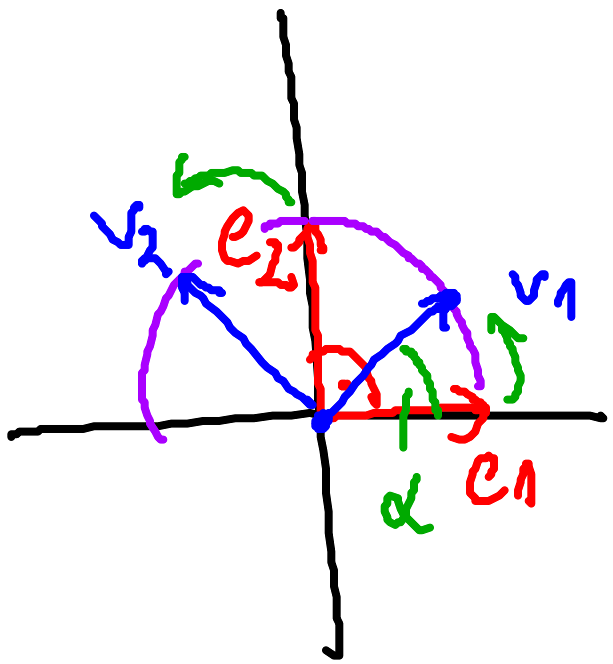
$$\binom{1}{4} + \mathrm{Kern}(f) = \left\{ \binom{x}{y} \;\middle|\; f\binom{x}{y} = f\binom{1}{4} \right\}$$

Einschub: Lineare Abb als geometrische Transform.

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x \cdot v_1 + y \cdot v_2 = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Die Bilder der Einheitsvektoren sind die Spalten der Matrix.



Bsp Drehung

(i) v_1, v_2 Länge 1

(ii) $v_1 \perp v_2$

(iii) v_2 gegen Uhrz. um $90°$ aus v_1

Vorschau auf LA II

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{x^2 + y^2}$$

Senkrechtstehen

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \perp \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Leftrightarrow$$

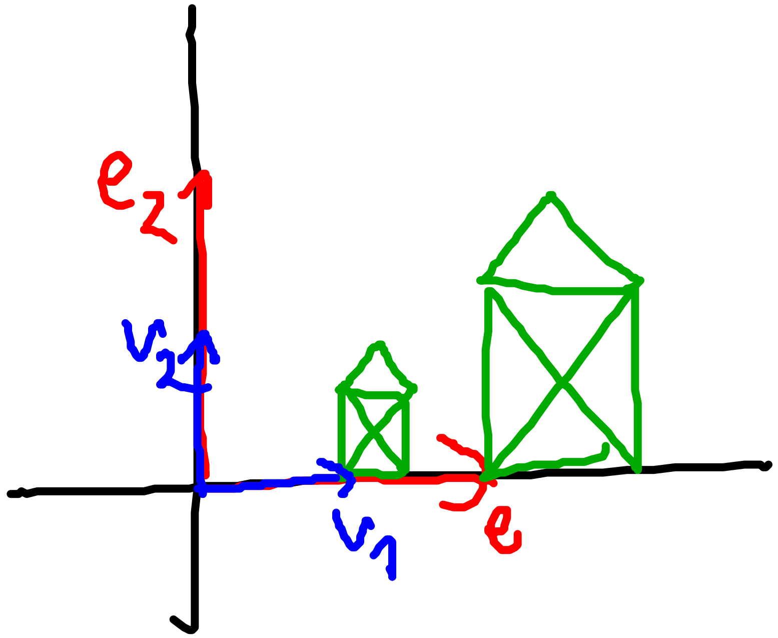
$$x_1 \cdot x_2 + y_1 y_2 = 0$$

Drehung am Nullpunkt um α

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Andere Transformationen!

Skalierung um den Nullpunkt



$$v_1 = \lambda \cdot e_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

$$v_2 = \lambda \cdot e_2 = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}$$

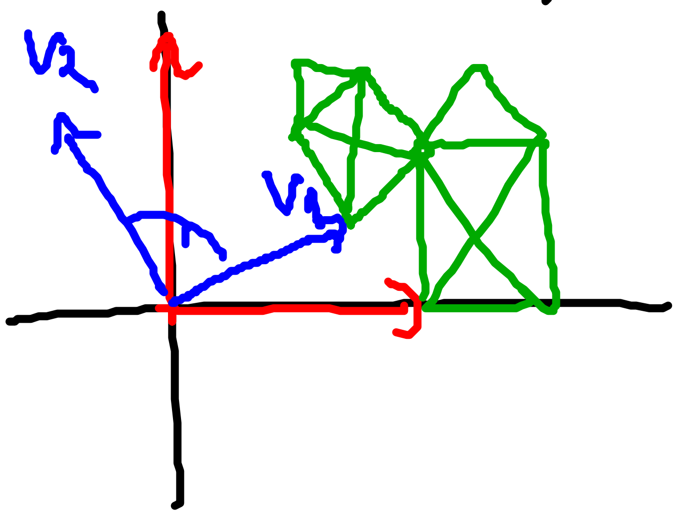
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Drehstreckung:

(i) $|v_1| = |v_2|$

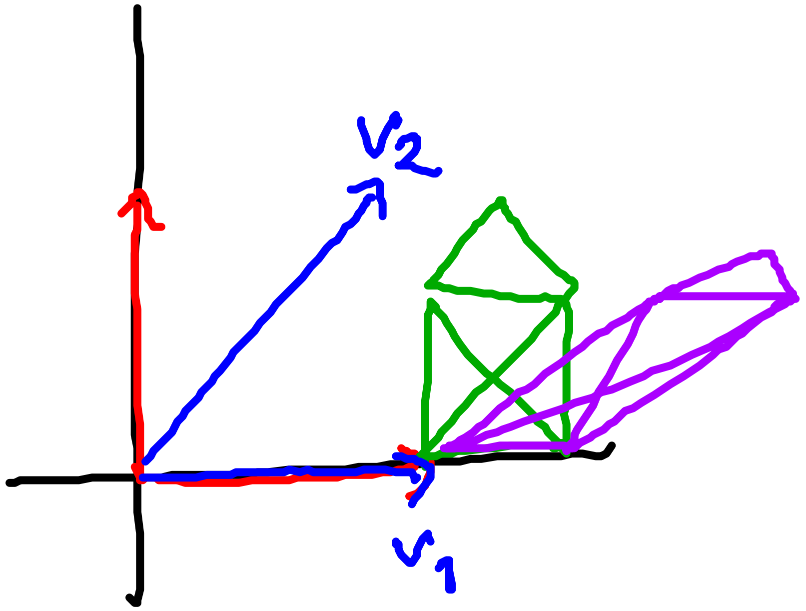
(ii) $v_1 \perp v_2$

(iii) Orientierung



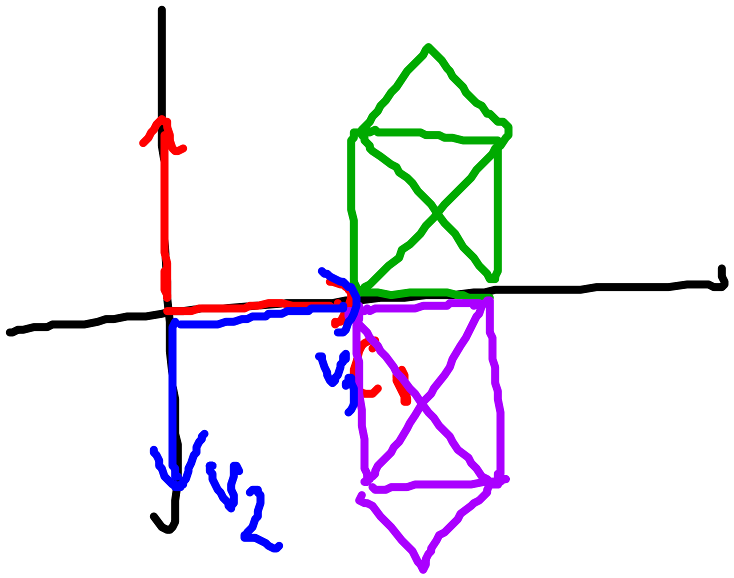
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Scherung entlang x Achse



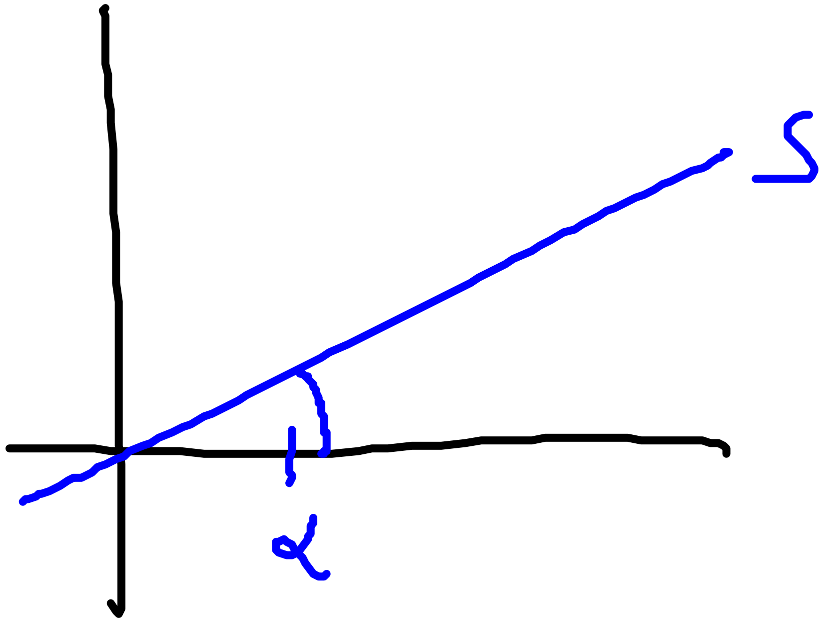
$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

Spiegelung an x Achse



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spiegelung um eine beliebige Achse:



$$D_\alpha \left(\begin{pmatrix}1&0\\0&-1\end{pmatrix} \left(D_{-\alpha} \cdot \begin{pmatrix}x\\y\end{pmatrix}\right)\right)$$

$$D_\alpha = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

$$= \left(D_\alpha \cdot \left(\begin{pmatrix}1&0\\0&-1\end{pmatrix} \cdot D_{-\alpha}\right)\right)\begin{pmatrix}x\\y\end{pmatrix}$$

Notwendig:
Matrizenmultiplikation

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{X}\left(\underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{Y}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a'x + b'y \\ c'x + d'y \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \Rightarrow \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}}$$

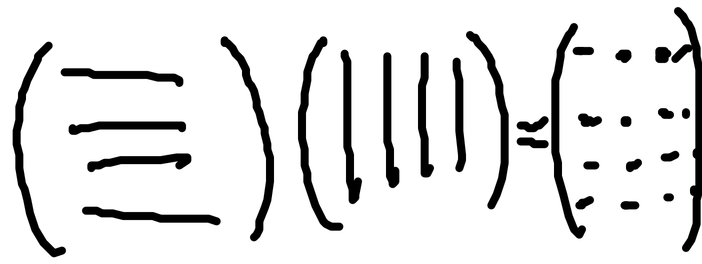
$$= \begin{pmatrix} aa'x + ab'y + bc'x + bd'y \\ ca'x + cb'y + dc'x + dd'y \end{pmatrix} = \underbrace{\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}}_{X \cdot Y}\begin{pmatrix} x \\ y \end{pmatrix}$$

Matrizenmultiplikation allgemein

$$\begin{array}{c|c} & 1 \quad\; n \quad\; 1 \\ \hline r & \begin{pmatrix} \cdots\cdots\; z_i \;\cdots\cdots \\ \\ \hfill f \end{pmatrix} \end{array} \quad \begin{array}{c|c} & 1 \quad k \quad 1 \\ \hline n & \begin{pmatrix} \vdots \\ s_j \\ \vdots \\ \hfill g \end{pmatrix} \end{array} = \begin{array}{c|c} & 1 \quad\; k \quad\; 1 \\ \hline r & \begin{pmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{pmatrix} \end{array} \begin{array}{c} i \\ \\ j \end{array}$$

$$z_i = (w_1, \ldots, w_n) \qquad s_j = (v_1, \ldots, v_n)$$

$$a_{ij} = \sum_{\ell=1}^{n} w_\ell \cdot v_\ell$$

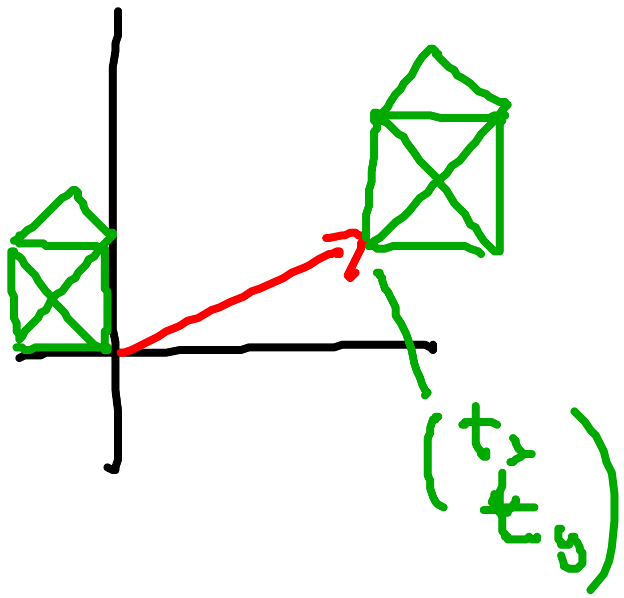


$g : \mathbb{K}^k \to \mathbb{K}^n$

$f : \mathbb{K}^n \to \mathbb{K}^r$

$f \circ g : \mathbb{K}^k \to \mathbb{K}^r$

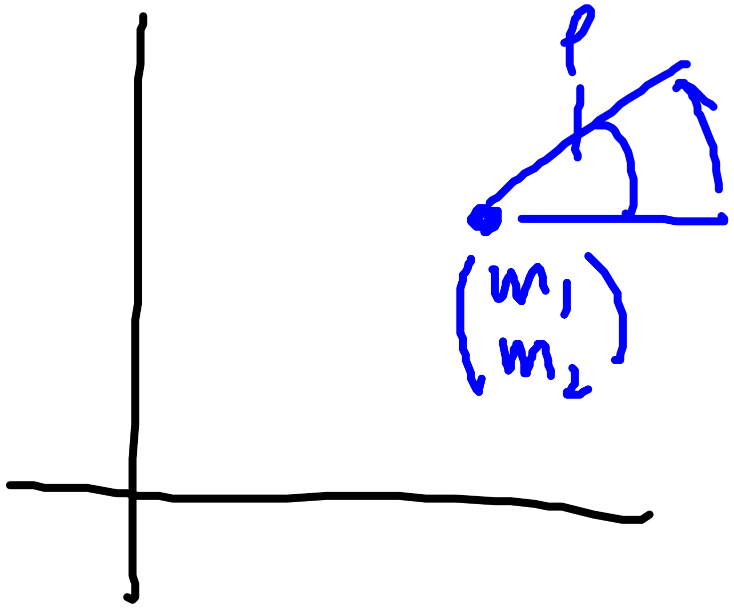
Translationen



$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Sind <u>keine</u> linearen Abb.

Drehung um beliebigen Punkt



$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto D_\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right) + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

Sind keine linearen Abb.