Anwendungen von Determinanten

Zuerst Crashcours Determinanten:

$\det : K^{n\cdot n} \to K$ (ordnet $n\times n$ Matrix Zahl zu)

$\det : (K^n)^n \to K$ (ordnet n n-dim Vektoren eine Zahl zu.)

Charakterisiert durch drei Eigenschaften

(i) linear in jeder Matrixspalte (multilinear)

(ii) Vertauschen zweier Spalten ändert Vorz. (alternierend)

(iii) $\det(E_n) = 1$ (normiert)

Durch (i)–(iii) eindeutig bestimmt.

$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = a\cdot d - b\cdot c$; $\det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = aei + dhc + gbf - ahf - dbi - gec$

$\det(A\cdot B) = \det(A)\cdot\det(B)$

Wichtige Aspekte:

→ Absolutbetrag

→ Fläche

→ Volumen

⋮

→ Vorzeichen

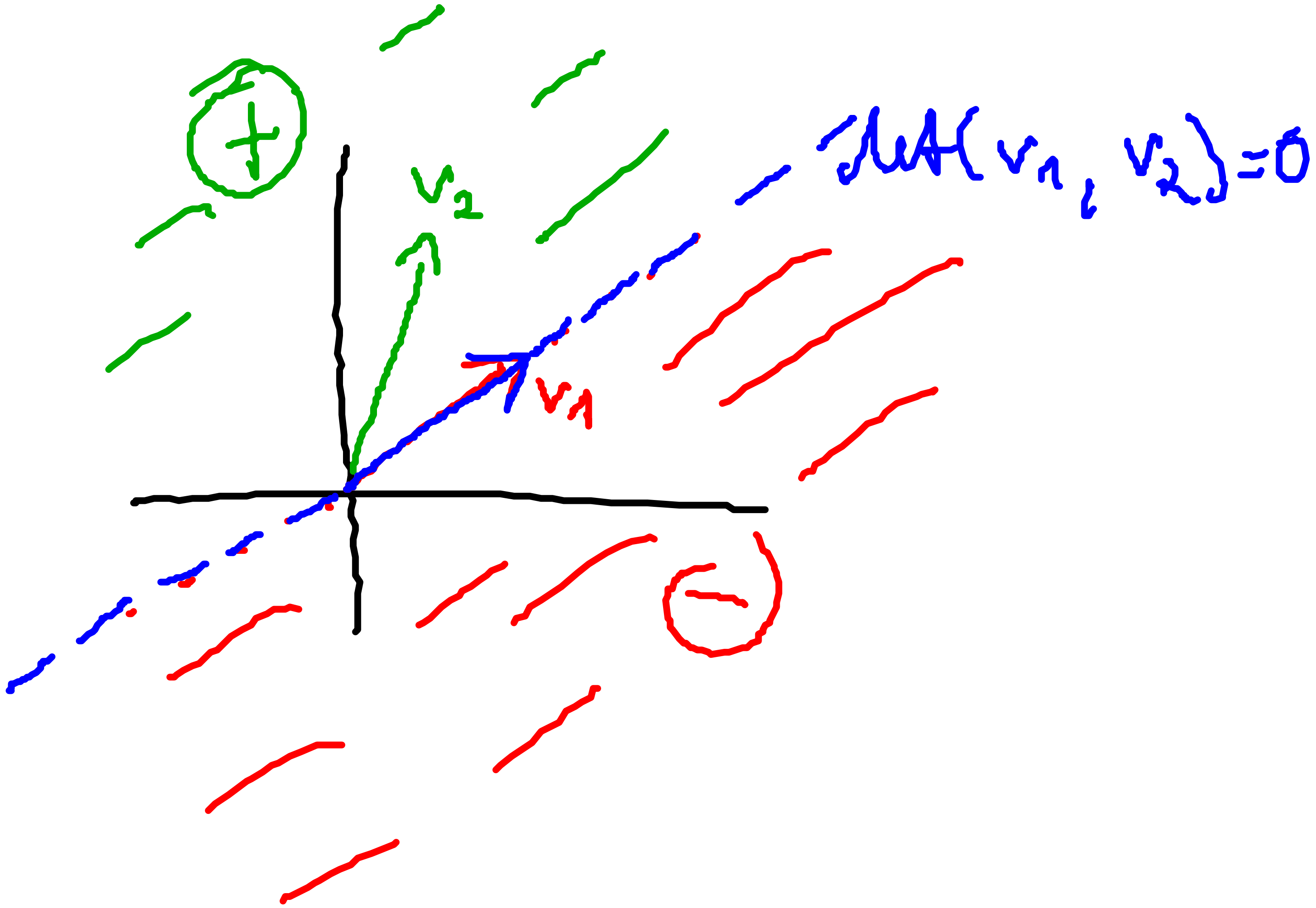
→ Relative Lage von Vektoren

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$$

$$\begin{array}{cc} v_1 & v_2 \end{array}$$

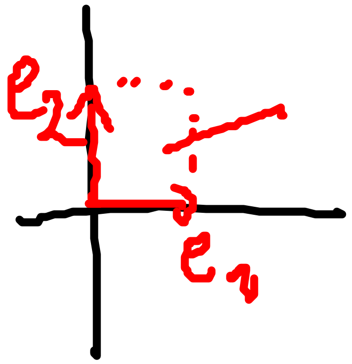
linear in v_2

$$\det\begin{pmatrix} a & \lambda a \\ b & \lambda b \end{pmatrix} = a \cdot \lambda b - b \cdot \lambda a = 0$$



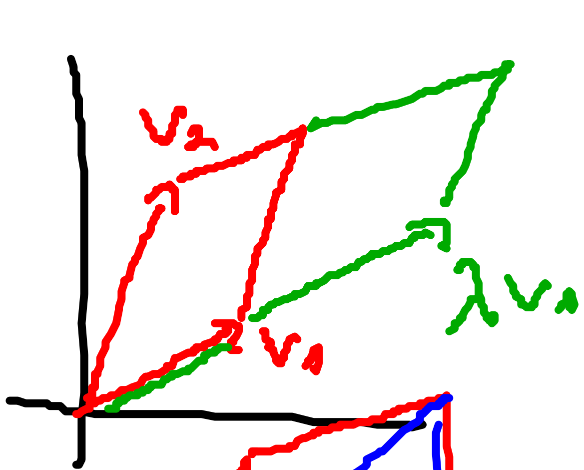
Orientierte Fläche eines Parallelogramms ist det

(i)

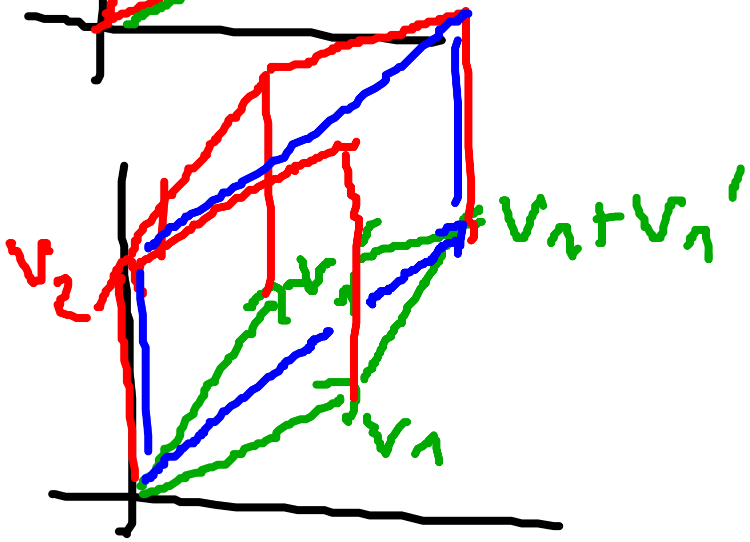


$\mathrm{area}(e_1, e_2) = 1$

(ii)



$\mathrm{area}(\lambda v_1, v_2) = \lambda\, \mathrm{area}(v_1, v_2)$



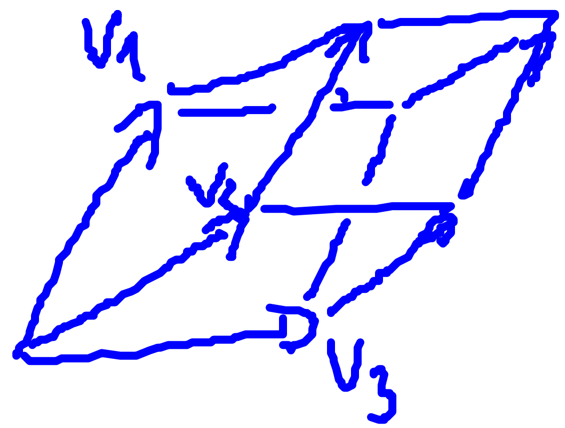
$\mathrm{area}(v_1 + v_1', v_2) =$

$\mathrm{area}(v_1, v_2) + \mathrm{area}(v_1', v_2)$

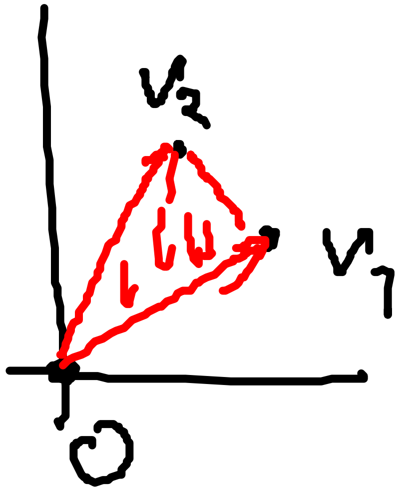
(iii) alternierend...

höhere Dimensionen:

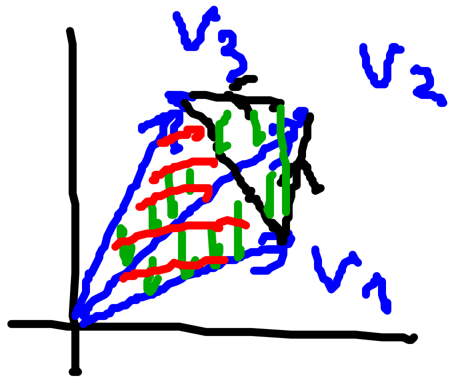
$\det(v_1, v_2, v_3)$ ist orientiertes Vol des von v_1, v_2, v_3 aufgesp. Parallelotops



Orientierte Fläche eines Dreiecks



$$\Delta(0, v_1, v_2) = \frac{1}{2}\det(v_1, v_2)$$



$$\Delta(v_1, v_2, v_3) = \frac{1}{2}\det(v_1\ v_2) + \frac{1}{2}\det(v_2\ v_3) + \frac{1}{2}\det(v_3\ v_1)$$

$+$

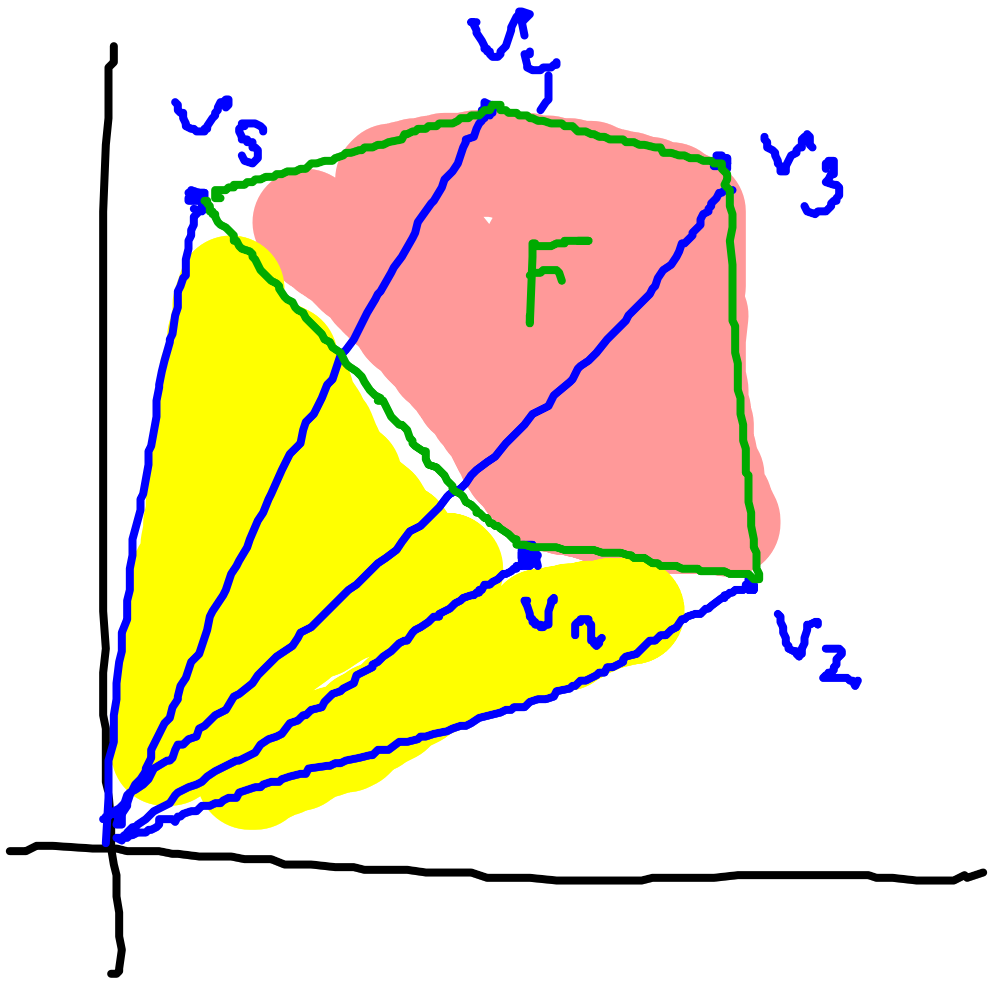
$+$

$-$

$\cdot 2$

$$\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} = x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3$$

Fläche eines beliebigen n-Ecks.



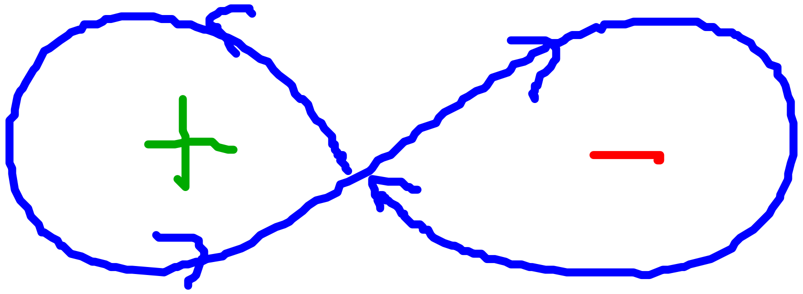
$$2 \cdot \mathrm{area}(F) =$$

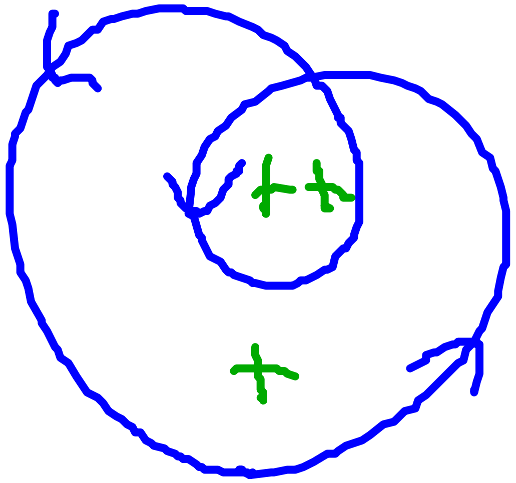
$$\det(v_1, v_2) + \det(v_2, v_3) + \det(v_3, v_4) + \det(v_4, v_5) + \det(v_5, v_1)$$

$$\mathrm{area}(v_1 \ldots v_n) = \frac{1}{2} \sum_{i=1}^{n} \det(v_i\, v_{i+1})$$

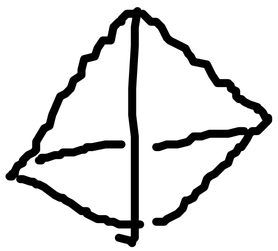
indizes mod n

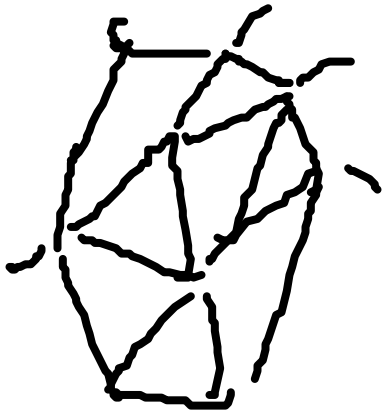
Allgemeine Polygone mit Windungszahl



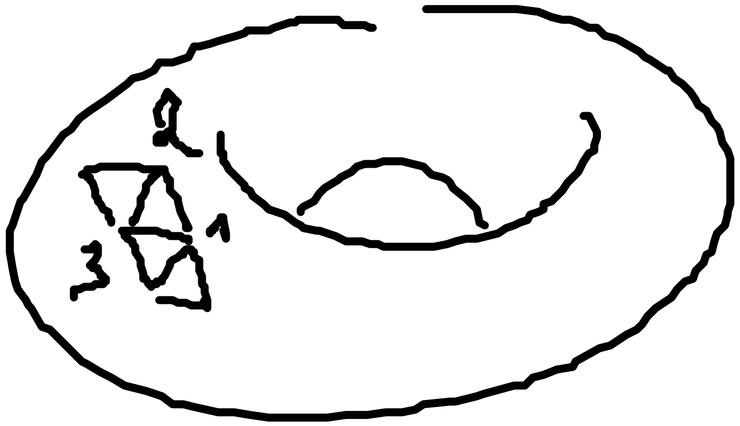


Höhere Dimensionen





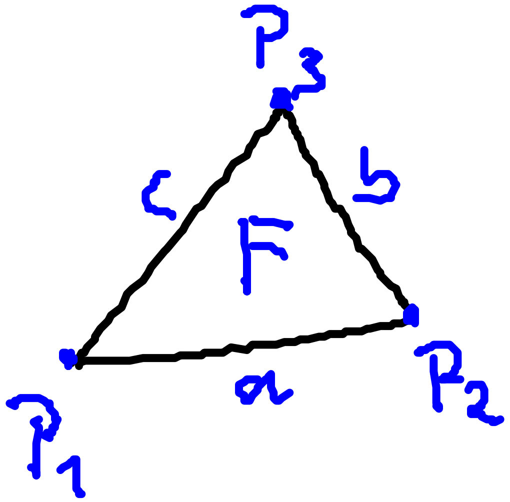
Triangulierte Ball



$$\sum_{(i,j,k)} \det(v_i\, v_j\, v_k)$$

aus Randtriang.

$\mathbb{R}^2$ Berechne Dreiecksfläche aus Seitenlängen



Herons Formel

$$F = \sqrt{s\cdot(s-a)(s-b)(s-c)}$$

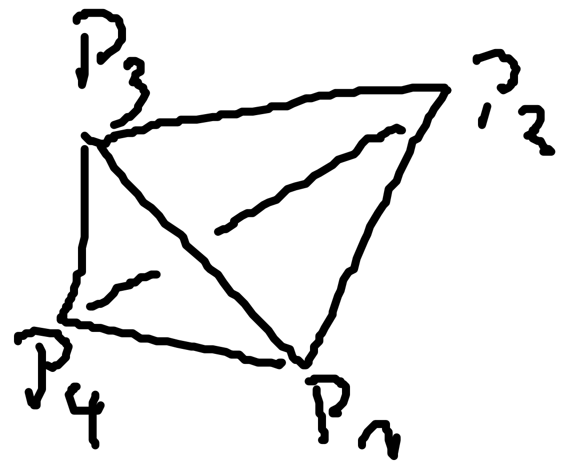
$$s = \tfrac{1}{2}\cdot(a+b+c)$$

$$d_{ij} = \|P_i - P_j\|^2$$

$$a = \sqrt{d_{12}}$$
$$b = \sqrt{d_{23}}$$
$$c = \sqrt{d_{31}}$$

$$\det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} \\ 1 & d_{12} & 0 & d_{23} \\ 1 & d_{13} & d_{23} & 0 \end{pmatrix} = -16\,F^2$$

$\mathbb{R}^3$ gegeben die Kantenlängen eines Tetraeders $\sqrt{d_{ij}}$

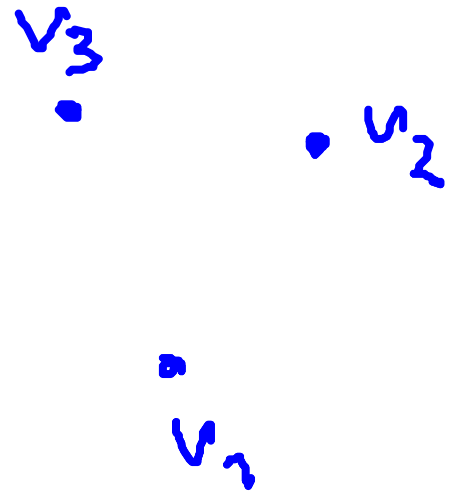


$$\det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} \\ 1 & d_{12} & 0 & d_{23} & d_{24} \\ 1 & d_{13} & d_{23} & 0 & d_{34} \\ 1 & d_{14} & d_{24} & d_{34} & 0 \end{pmatrix} = 288\,V$$

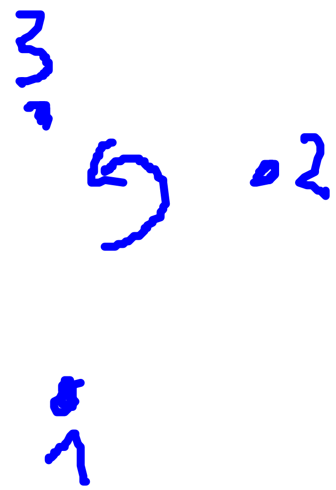
$$(-1)^{n+1}(n!)^2\cdot 2^n$$

Vorzeichen von Determinanten

$$\chi : E^3 \to \{-,0,+\}$$
$$E = \{1 \dots n\}$$

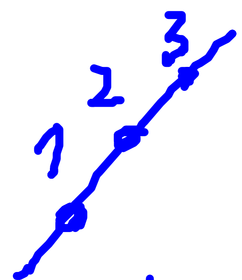


$$\operatorname{sign}\left(\det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}\right) = \chi(1,2,3)$$



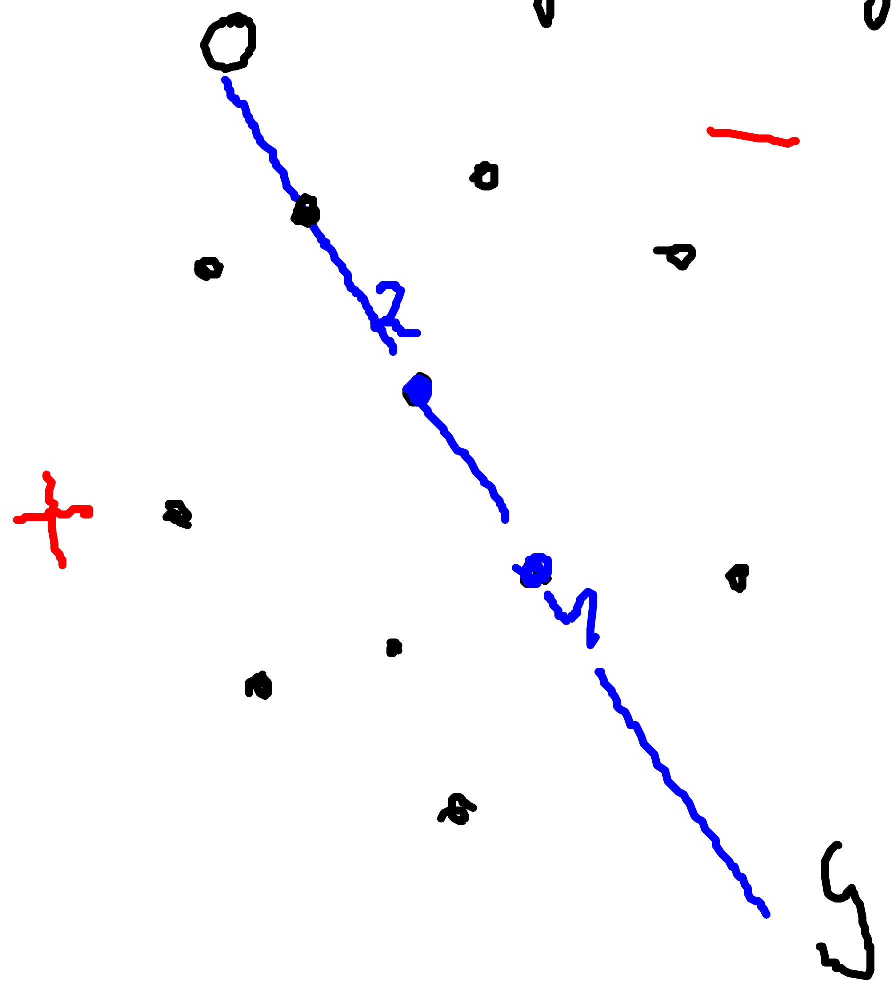
$\chi(1,2,3) = +$

$\chi(1,2,3) = -$



$\chi(1,2,3) = 0$

Relative Lage bzg Geraden



Alles auf g $= \{ i \mid \chi(1,2,i) = 0 \}$

Alles „links" von g $=$
$\{ i \mid \chi(1,2,i) = + \}$

Alles „rechts" von g $=$
$\{ i \mid \chi(1,2,i) = - \}$

z.B konvexe Hülle