Quotientenräume

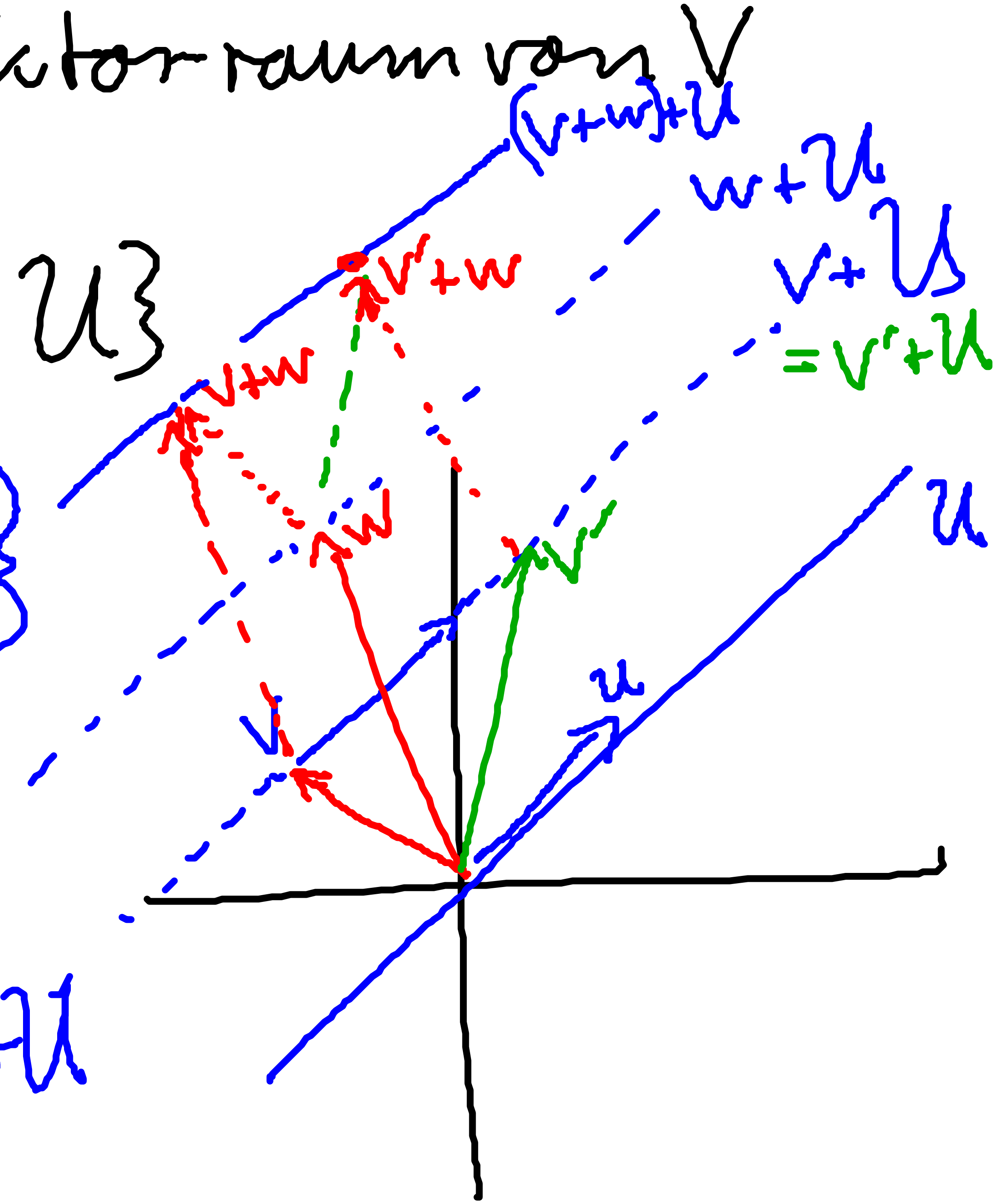
Sei $\mathcal{U}$ ein Untervektorraum von V

$v \in V$

$v + \mathcal{U} = \{ v + u \mid u \in \mathcal{U} \}$

$V/\mathcal{U} = \{ v + \mathcal{U} \mid v \in V \}$

Quotienten Raum

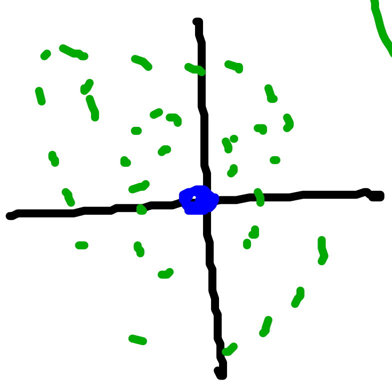


$(v+\mathcal{U}) \oplus (w+\mathcal{U}) = (v+w)+\mathcal{U}$

$\lambda \odot (v+\mathcal{U}) = (\lambda \cdot v)+\mathcal{U}$

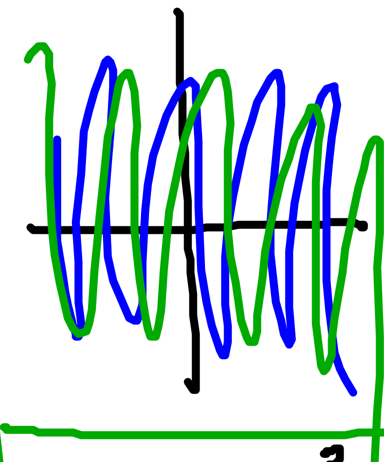
Verschiedene Dimensionen

$V = \mathbb{R}^2$ $\dim(U) = 0$



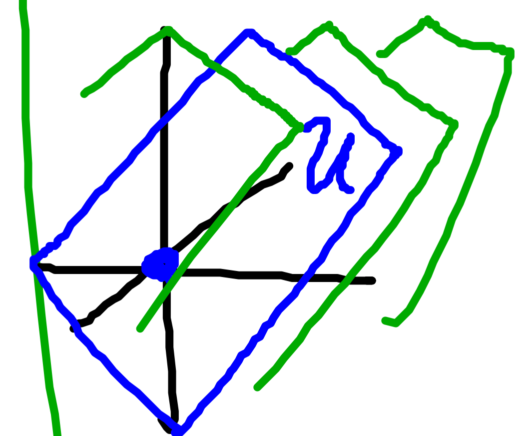
V/U = alle Punkte des $\mathbb{R}^2$

$V = \mathbb{R}^2$ $\dim(U) = 2$



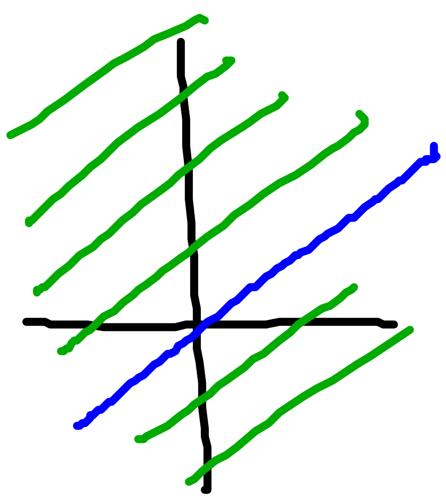
$V/U = \{\mathbb{R}^2\}$

$V = \mathbb{R}^3$ $\dim(U) = 2$



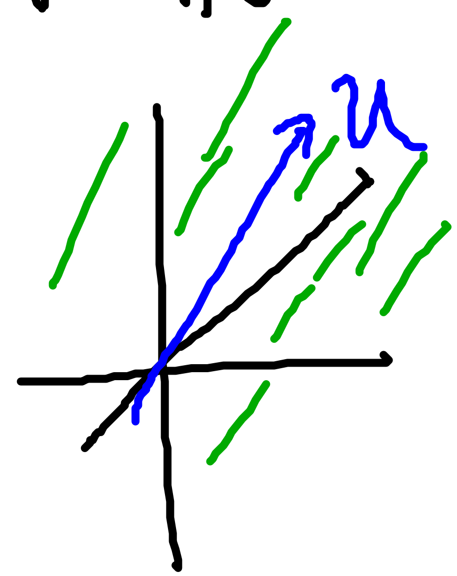
V/U = alle Ebenen parallel zu U

$V = \mathbb{R}^2$ $\dim(U) = 1$



V/U = alle Geraden parallel zu U

$V = \mathbb{R}^3$ $\dim(U) = 1$



V/U = alle Geraden parallel zu U

Isomorphiesatz

$f: V \to W$ ein Abb.

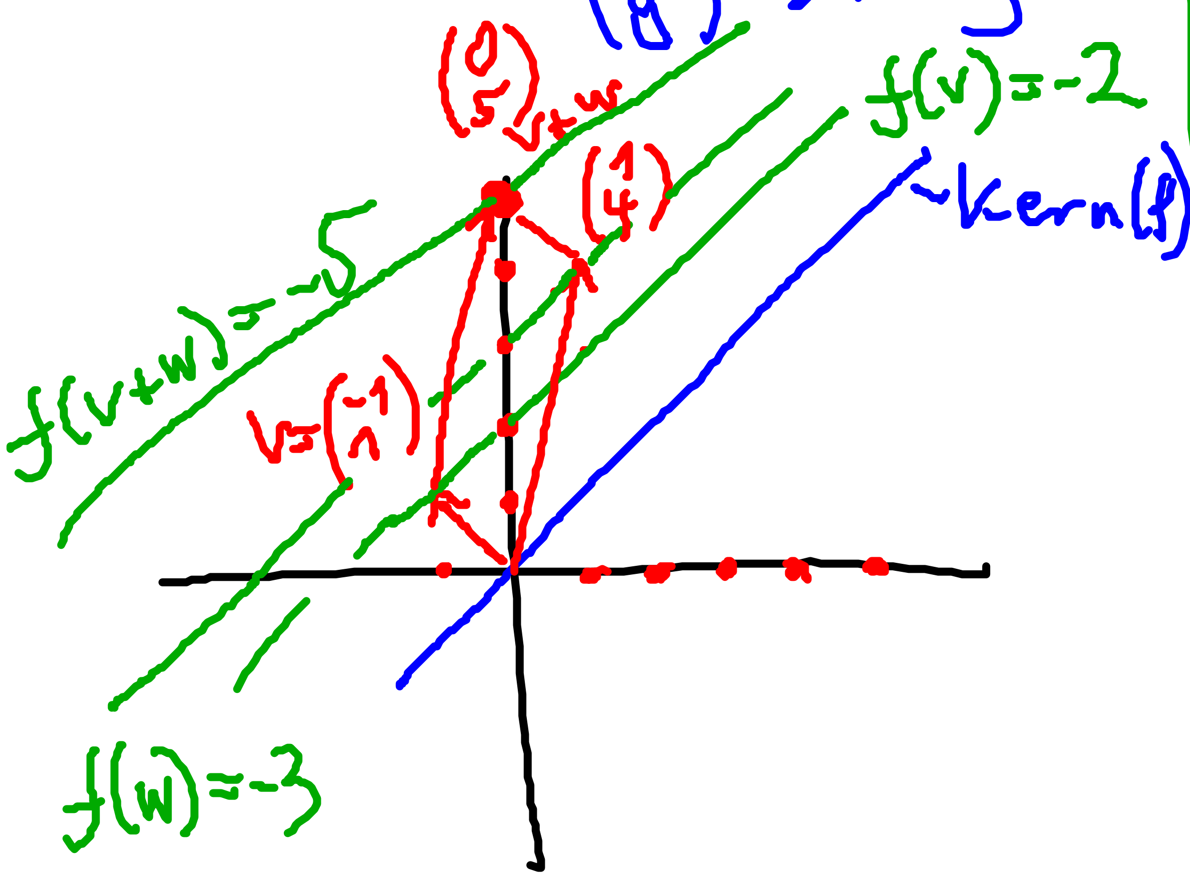
$$V/_{\mathrm{Kern}(f)} \sim \mathrm{Bild}(f)$$

$V/_{\mathrm{Kern}(f)} = \{v + \mathrm{Kern}(f) \mid v \in V\}$

$\mathrm{Bild}(f) = \{f(v) \mid v \in V\}$

Beispiel: $f: \mathbb{R}^2 \to \mathbb{R}$

$\begin{pmatrix} x \\ y \end{pmatrix} \to x - y$



$\Phi: V/_{\mathrm{Kern}(f)} \to \mathrm{Bild}(f)$

$v + \mathrm{Kern}(f) \mapsto f(v)$

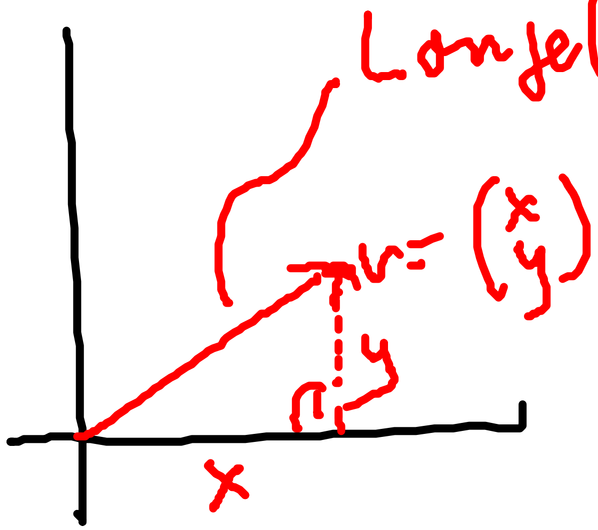
- Wohldefiniert
- bijektiv
- $\Phi(a+b) = \Phi(a) + \Phi(b)$
- $\Phi(\lambda \cdot a) = \lambda \cdot \Phi(a)$

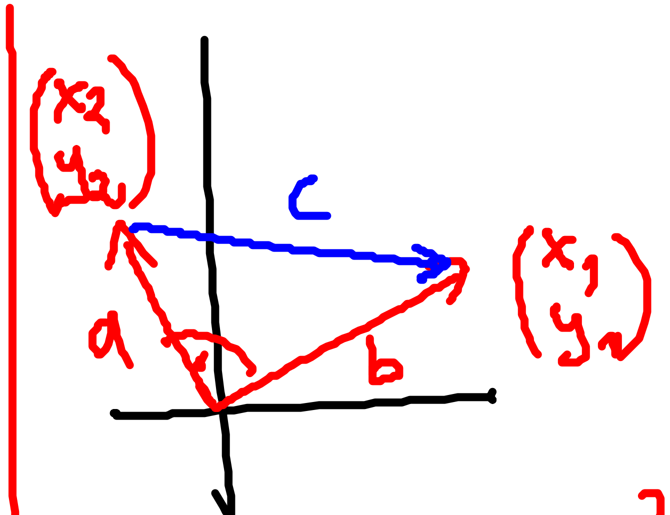
4.2 Lineare Abb. als geometrische Abb.

Geometrischer Exkurs: Länge, Senkrechtstehen (bezügl. Standardbasis des $\mathbb{R}^n$)

Im $\mathbb{R}^2$

$$\text{Länge}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2+y^2}$$





$$a = \text{Länge}\begin{pmatrix} x_3 \\ y_2 \end{pmatrix}$$

$$b = \text{Länge}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$c = \text{Länge}\begin{pmatrix} x_1 - x_2 \\ y_1 - x_2 \end{pmatrix}$$

$$a^2 + b^2 = c^2$$

$$\Leftrightarrow (x_1^2 + y_1^2) + (x_2^2 + y_2^2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\Leftrightarrow \boxed{x_1 \cdot x_2 + y_1 \cdot y_2 = 0}$$

Geometrische Abbildungen (linear)

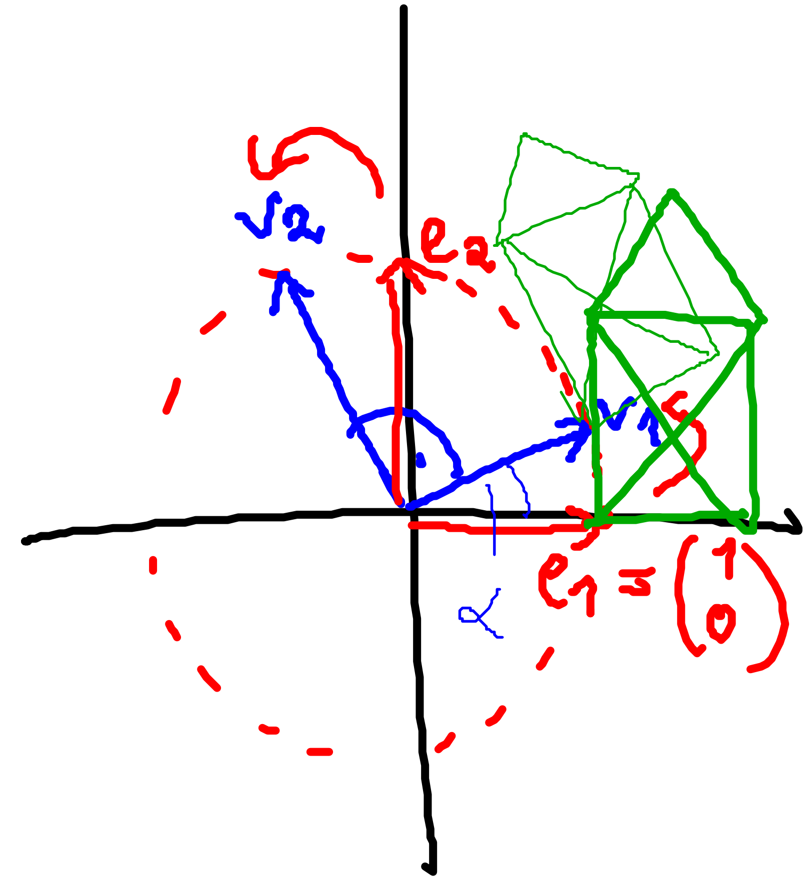
$f: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto v_1 x + v_2 y$

Beispiel 1:

v_1, v_2 beide Länge 1

v_1 senkrecht zu v_2
($v_1 \perp v_2$)



Drehung um α

$v_1 = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}$

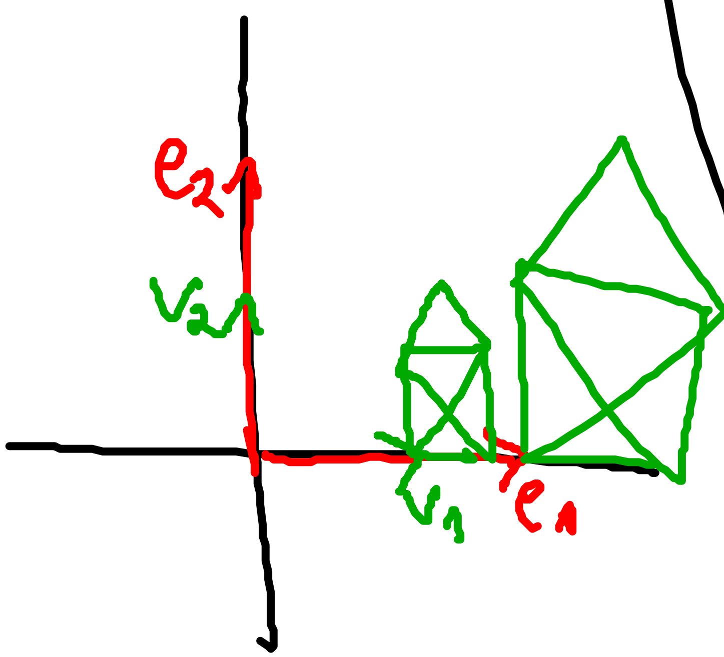
Weitere Beispiele

Translationen sind keine Linearen Abbildungen

Skalierung

$v_1 = \lambda e_1$

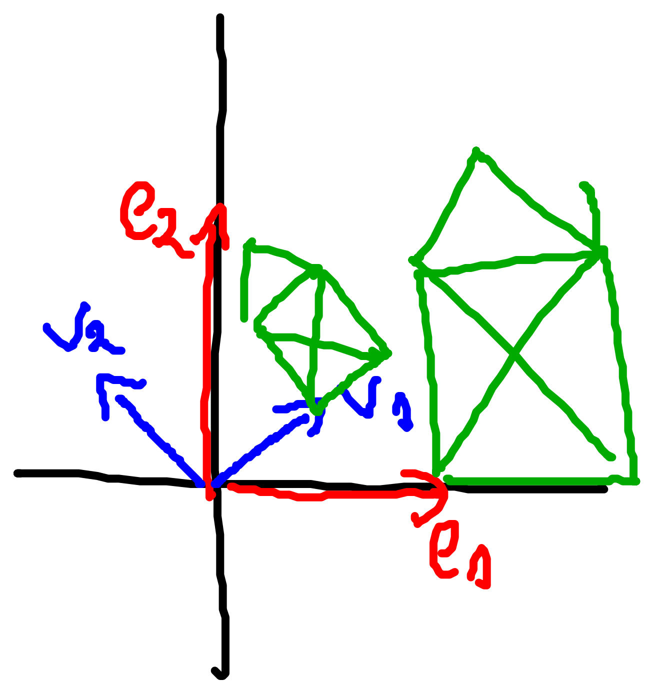
$v_2 = \lambda e_2$



Drehskalierung

v_1, v_2 haben gleiche Länge

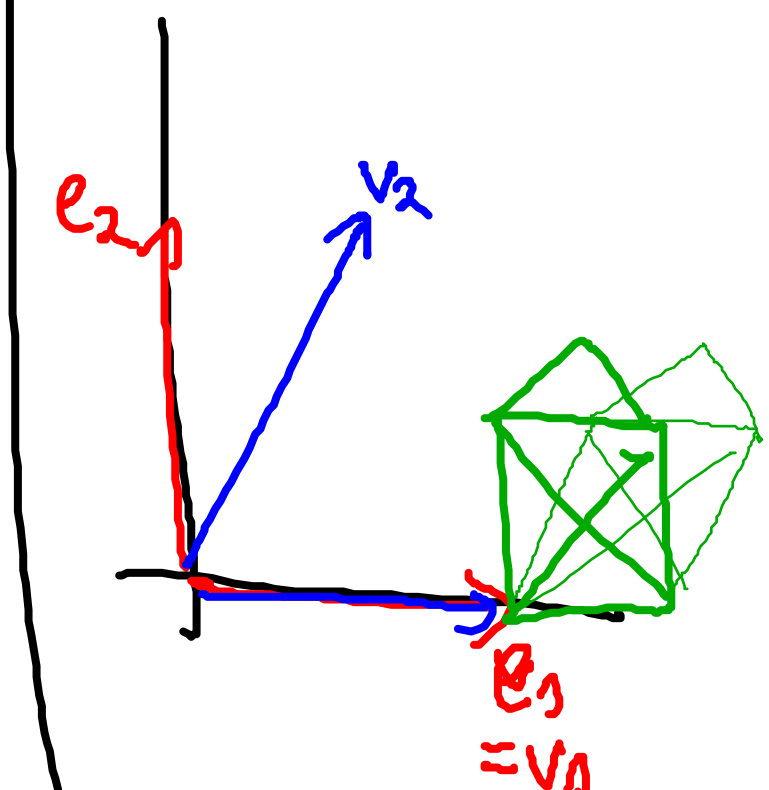
$v_1 \perp v_2$



Scherung

$v_1 = e_1$

$v_2 = e_2 + \lambda e_1$



Matrizenschreibweise

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto v_1 x + v_2 y$$

$$\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = v_1 \cdot x + v_2 y$$

$$v_1 = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\text{Matrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \text{Linearkomb der Spaltenvektoren}$$

Die Spalten der Matrix sind die Bilder der Einheitsvektoren

Allgemein: $f: K^n \to K^m$

$$f\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} = \lambda_1 v_1 + \dots + \lambda_n v_n \qquad v_1 \dots v_n \in K^m$$

$$= \lambda_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{1m}\end{pmatrix} + \dots + \lambda_n \begin{pmatrix} v_{n1} \\ \vdots \\ v_{nm}\end{pmatrix}$$

$$=: \begin{array}{cc} & n \\ m & \begin{pmatrix} v_{11} & & v_{n1} \\ \vdots & & \vdots \\ v_{1m} & & v_{nm}\end{pmatrix}\end{array} \cdot \begin{array}{cc} \begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} & n\end{array}$$

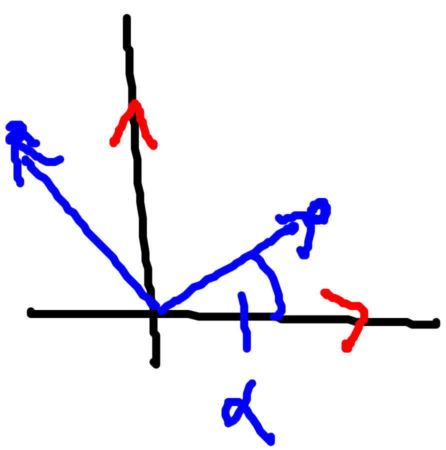
$m \times n$ Matrix

Beispiele:

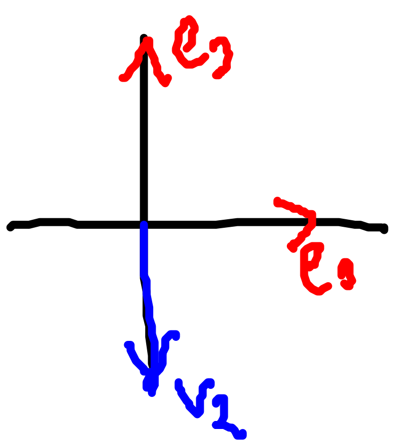
Skalierung: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda \cdot x \\ \lambda \cdot y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

Drehung um Winkel α:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

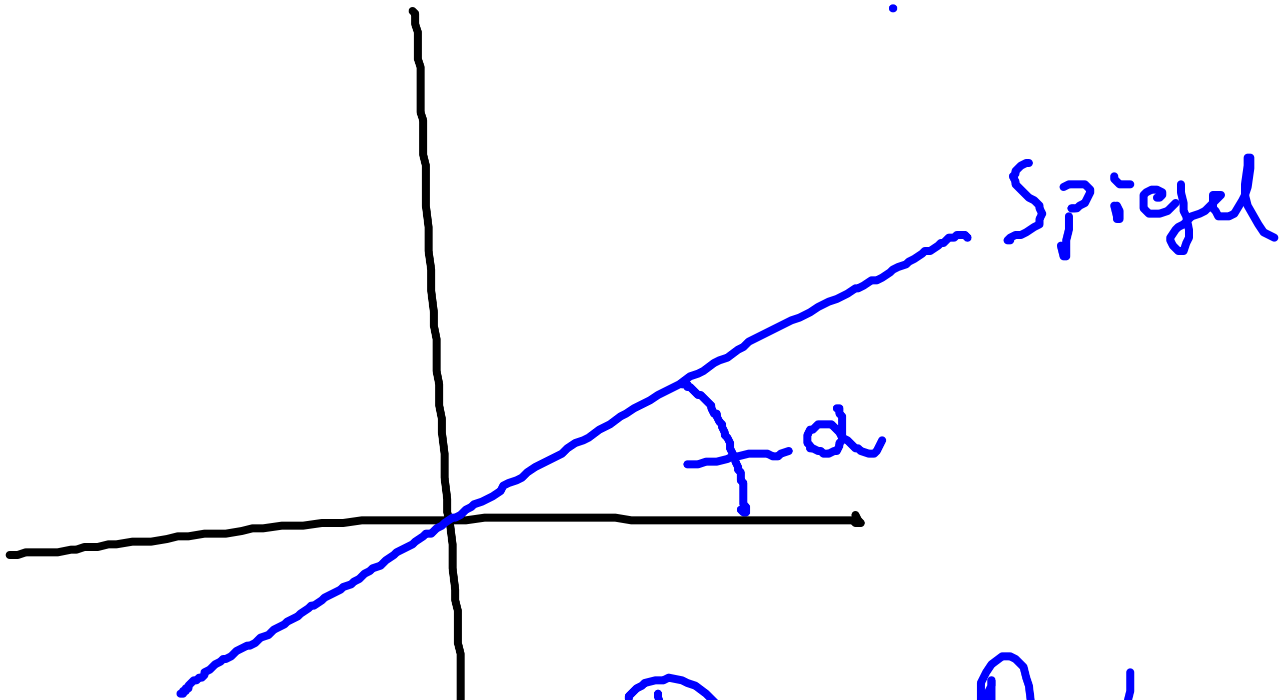


Spiegelung an x-Achse



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Allgemeine Spiegelung an Gerade durch O



$D_\alpha :=$ Drehung um α

$$D_\alpha \cdot \left(S_x \cdot \left(D_{-\alpha} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right) \right)$$

$$\underbrace{\left(D_\alpha \cdot S_x \cdot D_{-\alpha} \right)}_{2\times 2 \text{ Matrix}} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$