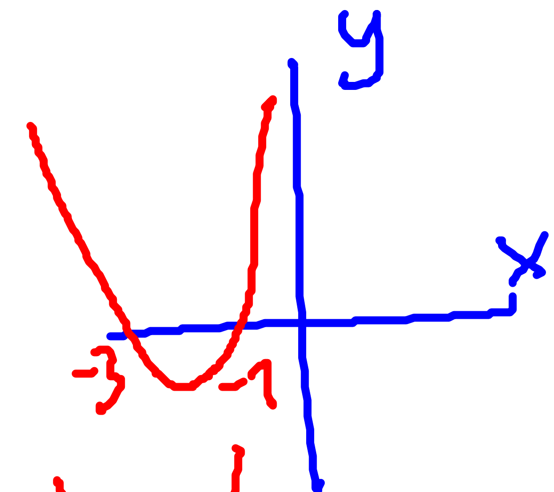
Komplexe Zahlen

Motivation: Nullstellen eines Polynoms berechnen.

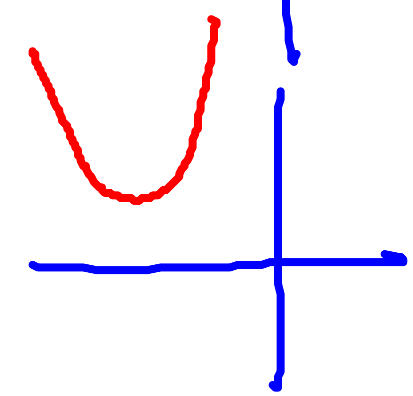
Bsp: Suche x mit $x^2+4x+3=0$

$$x_{1/2} = -2 \pm \sqrt{4-3} = -2 \pm 1$$

$$x_1 = -3, \quad x_2 = -1$$



Suche x mit $x^2+4x+5=0$



$$x_{1/2} = -2 \pm \sqrt{4-5} = -2 \pm \sqrt{-1}$$

Keine reellen Lösungen

Sei i eine „neue" Zahl mit $i^2 = -1$

$\sqrt{-1}$ „=" i

$$x_{1/2} = -2 \pm i$$

$x_1 = -2+i$
$x_2 = -2-i$

„Lösungen" der Gleichung

Probe $x^2+4x+5=0$

$$(-2+i)^2 + 4(-2+i)+5 =$$

$$\boxed{4}\ \boxed{-4i}\ \overset{i^2}{\boxed{-1}}\ \boxed{-8}\ \boxed{+4i}\ \boxed{+5}$$

$$0+0i = 0 \checkmark$$

Komplexe Zahlen:

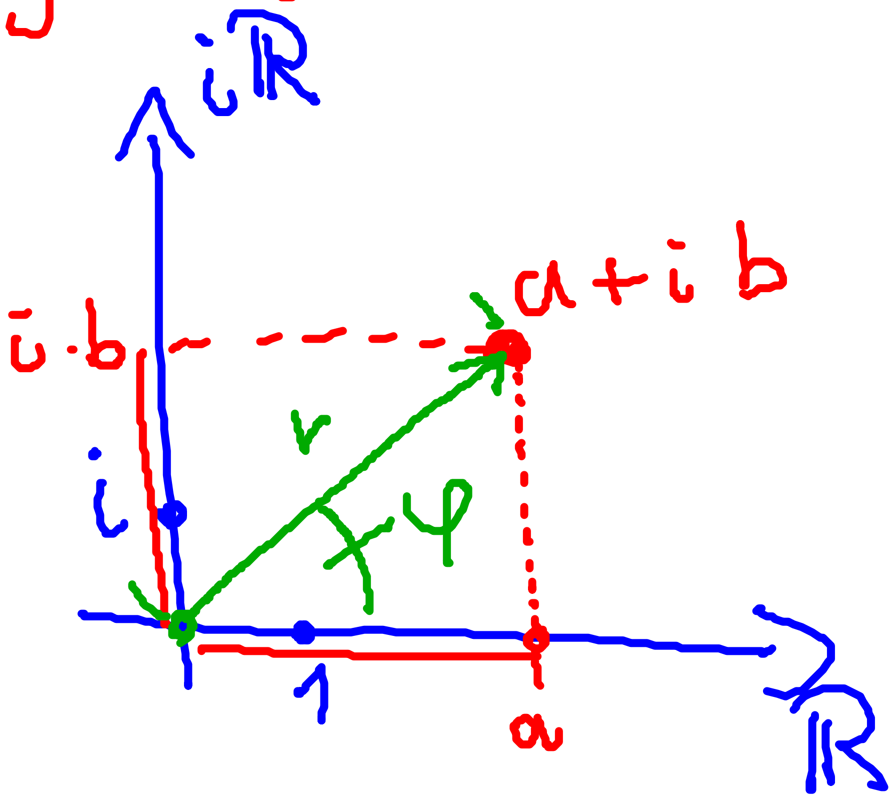
Zahlen der Form $\underbrace{\boxed{a}}_{\text{Realteil}} + \underbrace{\boxed{i\,b}}_{\text{Imaginärteil}}$, $a, b \in \mathbb{R}$

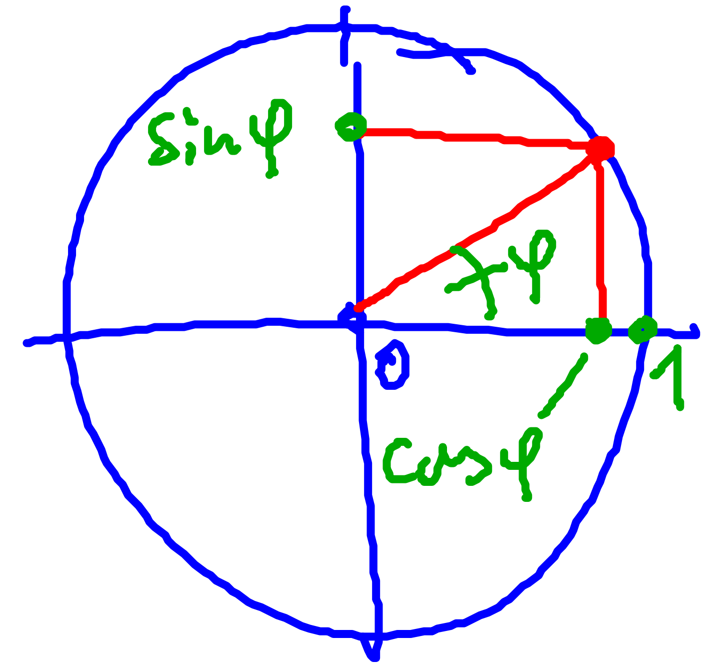
$\mathbb{C} = \mathbb{R} + i\mathbb{R}$ Zusätzliche Regel $i \cdot i = -1$

$$(a_1 + i b_1) + (a_2 + i b_2)$$
$$= (a_1 + a_2) + (b_1 + b_2) i$$

$$(a_1 + i b_1) \cdot (a_2 + i b_2)$$
$$= a_1 \cdot a_2 + a_1 \cdot i b_2 + i b_1 \cdot a_2 + i^2 b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$$

Geometrische Vorstellung





$$r = \sqrt{a^2+b^2}$$

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

$$a+ib = r\cdot(\cos\varphi + i\sin\varphi) = r\cdot e^{i\varphi}$$

$$e = 2{,}71828\ldots$$

Aus der Analysis: Potenzreihen

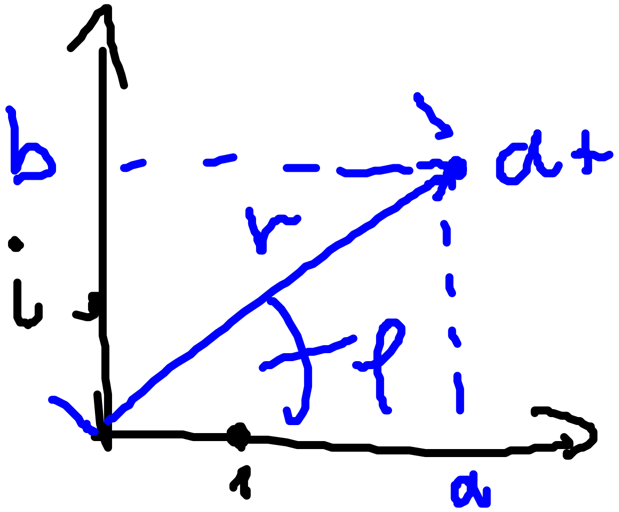
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$i\sin(x) = ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

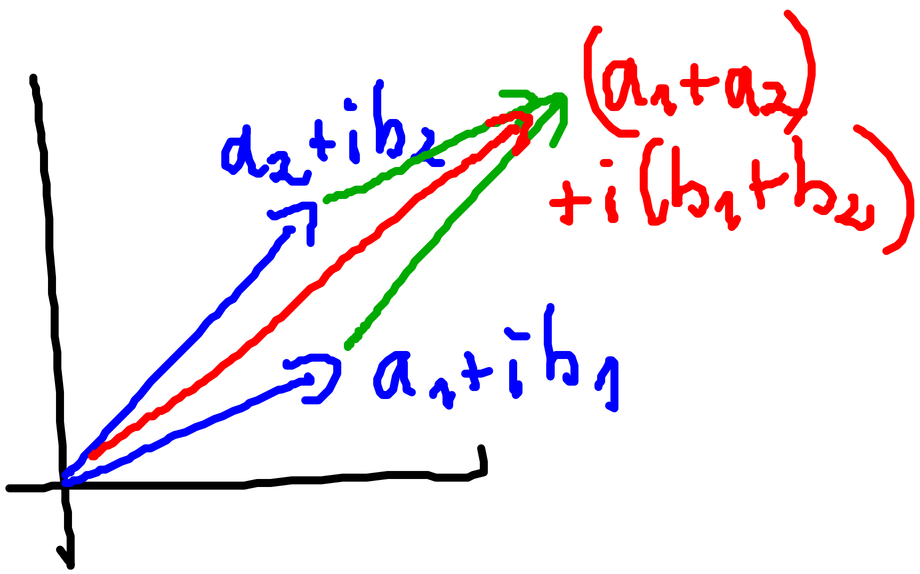
$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} \cdots$$

Also: $\boxed{e^{ix} = \cos(x) + i\sin(x)}$

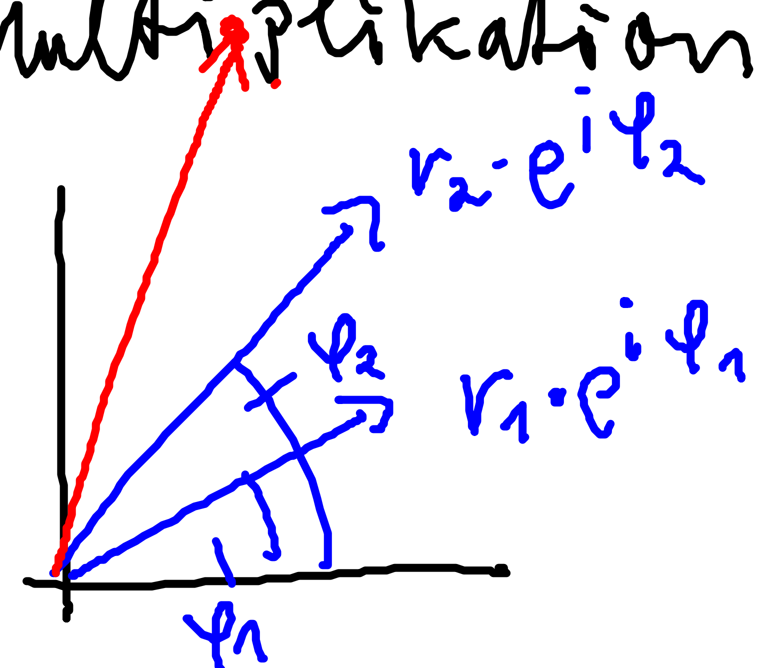


$a + ib = r \cdot e^{i\varphi}$

Addition



Multiplikation



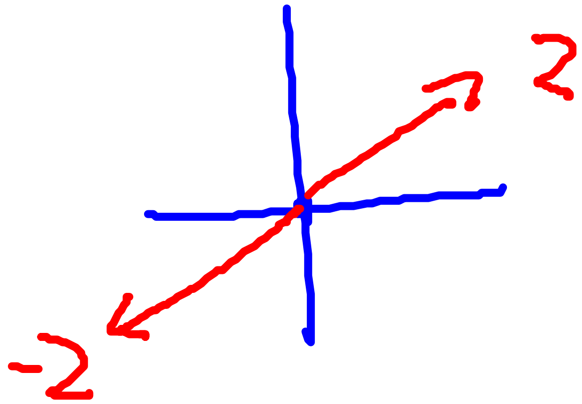
$$r_1 \cdot e^{i\varphi_1} \cdot r_2 \cdot e^{i\varphi_2}$$

$$= \underline{r_1 \cdot r_2 \cdot e^{i(\varphi_1+\varphi_2)}}$$

$(\mathbb{C},+,\cdot)$ ist Körper

$(\mathbb{C},+)$ ist kommutative Gruppe

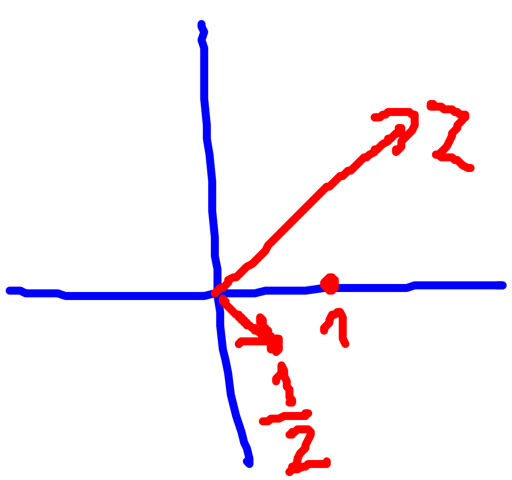
neutrales El $= 0$



Inverse zu $a+ib$ ist $-a-ib$

$(\mathbb{C}-\{0\},\cdot)$ ist kommutative Gruppe

neutrales El $= 1$



Inverses zu $z = r\cdot e^{i\varphi}$ ist $\frac{1}{r}\cdot e^{-i\varphi}$

Geschichtlicher Hintergrund:

Gleichungen Dritten Grades

„Löse" $x^3 + ax^2 + bx + c = 0$

$\rightsquigarrow$ gib eine Formel für die Lösungen an

Analog zur p,q-Formel

$$x^2 + px + q \Rightarrow x_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Literaturtip:

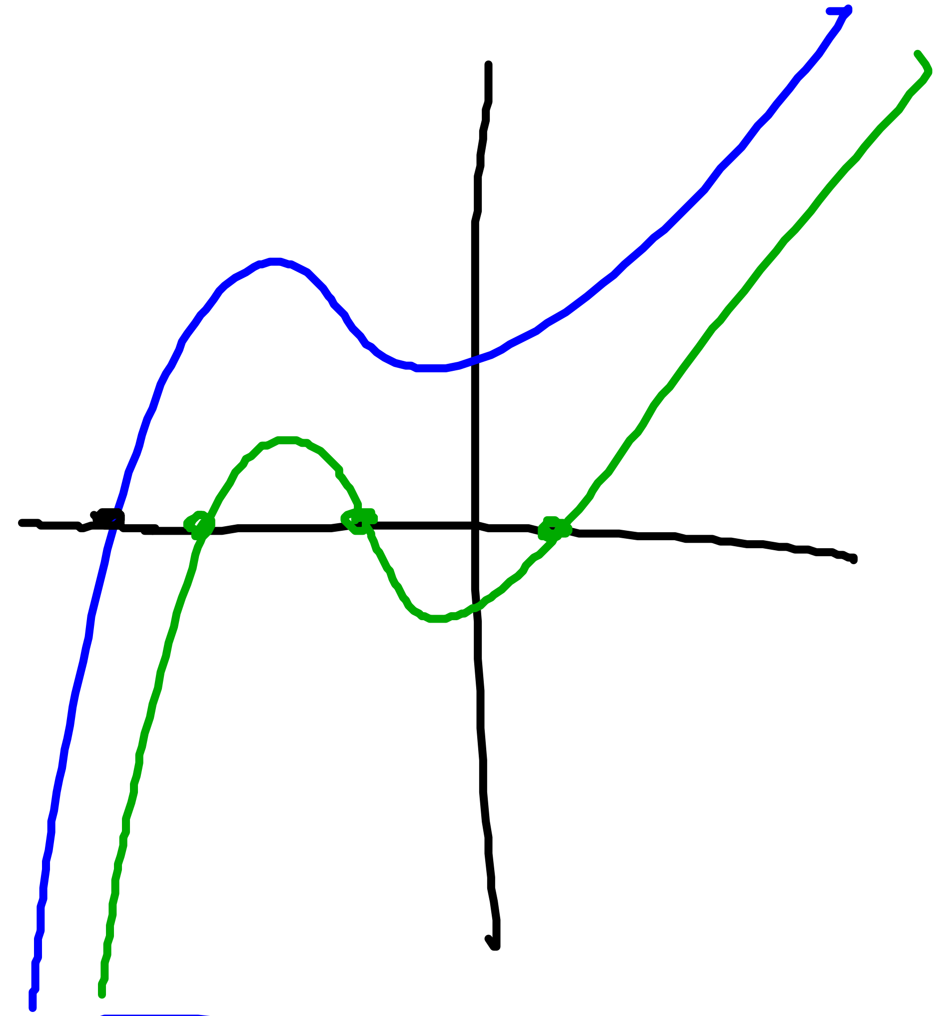
Yaglom: Felix Klein und Sophus Lie
Die Evolution des Symmetriebegriffs im 19. Jahrhundert
(Birkhäuser)

Kubische Gleichungen

$$x^3 + ax^2 + bx + c = 0$$

$\downarrow \quad x = z - \frac{a}{3}$

$$z^3 + pz + q = 0$$



$$z = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt[2]{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Tartaglia / del Ferro
Formel

$$x^3 - 24x - 72 = 0$$

$$x = \sqrt[3]{36 + \sqrt{36^2 + (-8)^3}} + \sqrt[3]{36 - \sqrt{36^2 + (-8)^3}}$$

$$= \sqrt[3]{36 + \sqrt{784}} + \sqrt[3]{36 - \sqrt{784}}$$

$$= \sqrt[3]{36 + 28} + \sqrt[3]{36 - 28}$$

$$= \sqrt[3]{64} + \sqrt[3]{8} = 4 + 2 = \underline{\underline{6}}$$

$$x^3 - 15x - 4 = 0$$

$$\sqrt[3]{2+\sqrt[2]{2^2-5^3}} + \sqrt[3]{2-\sqrt{2^2-5^3}}$$

$$= \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$$

$$= \sqrt[3]{2+i\cdot 11} + \sqrt[3]{2-i\cdot 11}$$

$$= (2+i) + (2-i) = 4$$

$$2+i = \sqrt[3]{2+i\cdot 11}$$

$$(2+i)^2 = 4+4i-1$$
$$= 3+4i$$
$$(3+4i)\cdot(2+i) =$$
$$6+11i-4 = \underline{\underline{2+11i}}$$